Stochastic Optimal Control Problems Related to Martingale Optimal Transport

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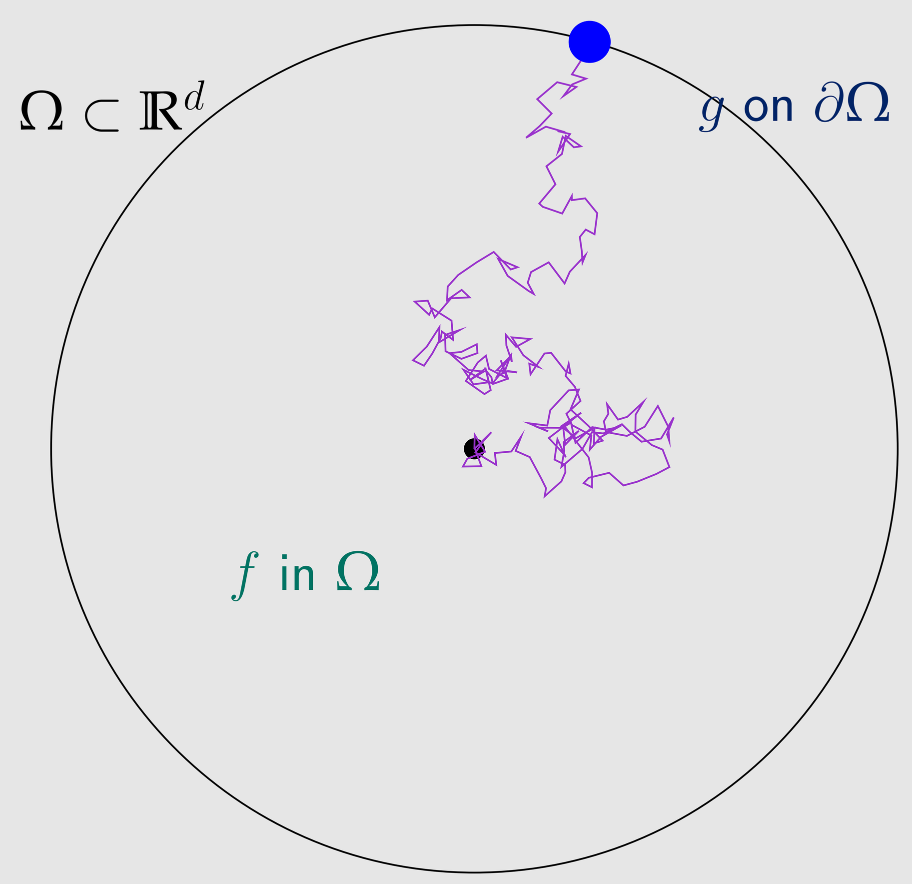
Stochastic Optimal Control

In a stochastic optimal control problem, we wish to minimise some quantity which depends on a random process. The random process itself depends on a control process, which we are allowed to choose in order to get the optimal behaviour. Problems of this type are treated in the textbooks [1, 2], for example.

Problem Setup

- We consider a random process on a compact domain $\Omega \subseteq \mathbb{R}^d$, which runs until it hits the boundary.
- We wish to find the **value function**:
$$v(x) = \inf_{\sigma \in \mathcal{U}} \mathbb{E}^x \left[\int_0^\tau f(X_s^\sigma)\,\mathrm{d}s + g(X_\tau^\sigma) \right],$$
where
$$\tau = \inf\{t \geq 0 : X_t^\sigma \in \partial\Omega\}.$$
- The below diagram shows a realisation of such a random process.

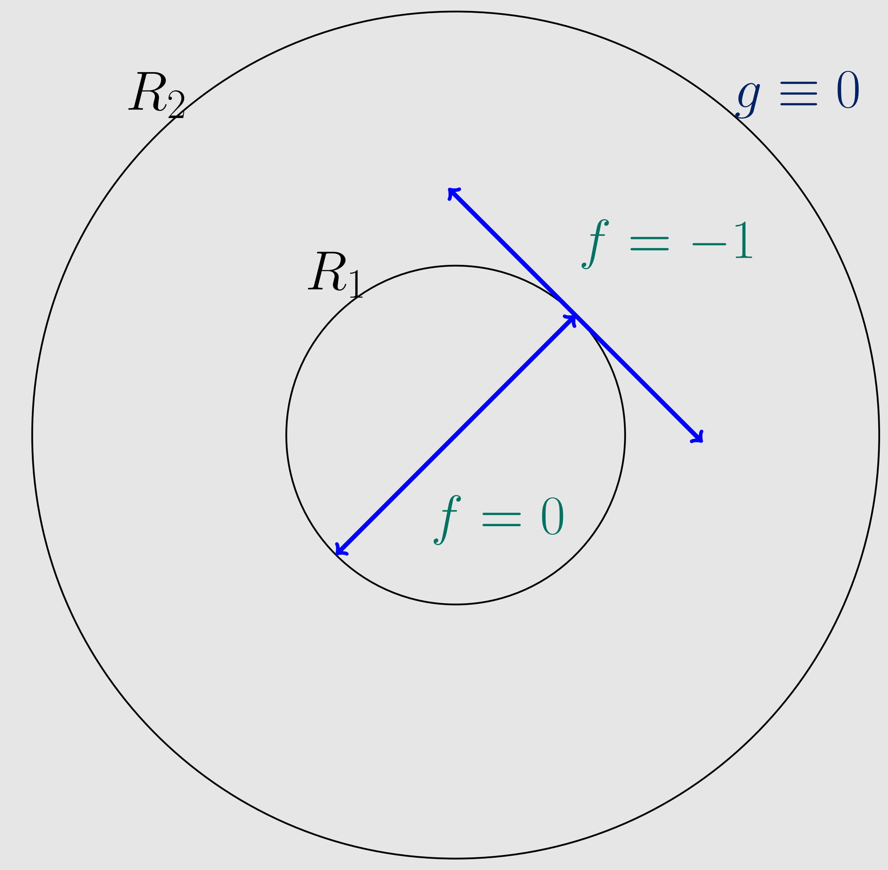


The Control Set

- We want to optimise over the set of **martingales** X which have unit speed. We think of martingales as processes whose value is expected to stay the same on average over time.
- We can write such a process X as
$$\mathrm{d}X_t^\sigma = \sigma_t\,\mathrm{d}B_t,$$
where
$$\sigma_t \in U := \left\{\sigma : \mathrm{Tr}(\sigma\sigma^\top) = 1\right\},$$
and B is a Brownian motion.
- Therefore the set of controls $\mathcal{U}$ should be some set of processes which take values in U.
- This is a natural analogue of **Brownian motion** in higher dimensions.
- A related choice for U, as studied in [3], is
$$\tilde{U} := \left\{\sigma : \det(\sigma\sigma^\top) \geq \frac{1}{d^d}\right\}.$$
- In some cases, U and $\tilde{U}$ give equivalent optimisation problems.

Example

- Consider the following example with $X = (X^{(1)}, X^{(2)}) \in \mathbb{R}^2$.
- Find
$$v(x) = \inf_{\sigma \in \mathcal{U}} \left\{ \mathbb{E}^x \left[-\int_0^\tau \mathbf{1}_{|X_s^\sigma| > R_1} \right] \right\},$$
where $\tau = \inf\{t \geq 0 : |X_t^\sigma| = R_2\}$.



- The optimal strategy should spend as much time as possible in the outer ring.
- This can be achieved by moving tangentially to the inner circle, as shown in the diagram above.
- We can write down the value function explicitly as
$$v(x) = \begin{cases} R_1^2 - R_2^2, & |x| \leq R_1, \\ x^2 - R_2^2, & R_1 < |x| \leq R_2. \end{cases}$$

Dynamic Programming Principle

- The key idea in solving such a problem is that, if the process follows a sub-optimal strategy, its total expected value increases over time.
- We say that v satisfies a dynamic programming principle if
$$v(X_t^\sigma) + \int_0^t f(X_s^\sigma)\,\mathrm{d}s \quad \text{is a } \textbf{submartingale}$$
for any σ (i.e. this quantity has an upward trend), and a martingale for the optimal control.
- From this, Itô's formula gives us a PDE formulation of the problem (see [1, 2]).

Hamilton-Jacobi-Bellman Equation

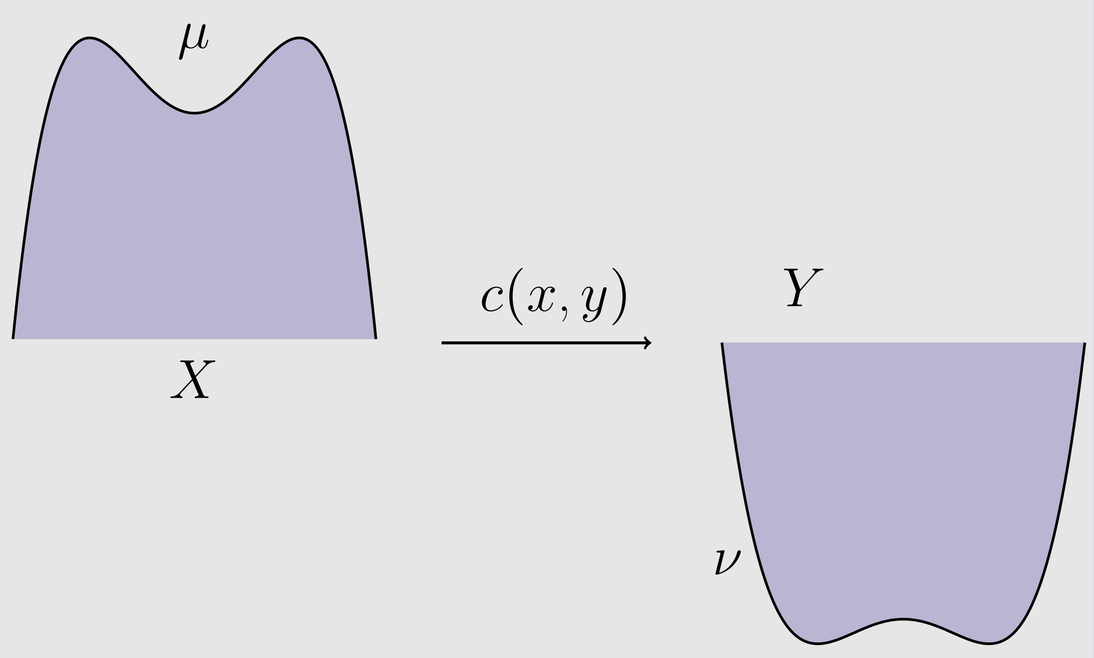
- The value function v should satisfy the following **Hamilton-Jacobi-Bellman** (HJB) equation:
$$\begin{cases} \frac{1}{2}\inf_{\sigma \in \mathcal{U}}\left\{\mathrm{Tr}(\sigma\sigma^\top D^2 v)\right\} + f = 0 & \text{in} \quad \Omega, \\ v = g & \text{on} \quad \partial\Omega. \end{cases}$$
- As in our example, v is not usually smooth.
- We interpret the PDE in the **viscosity** sense, a weak form introduced in [4].
- Under some conditions, we have a comparison principle, and v is the unique viscosity solution.

The Monge-Ampère Equation

- The HJB equation is a Monge-Ampère equation when we optimise over the set $\tilde{U}$.
- We see this by using the algebraic identity:
$$\inf\left\{\mathrm{Tr}(AB) : B \text{ spd}, \det(B) \geq \frac{1}{d^d}\right\} = \det(A)^{\frac{1}{d}},$$
for any positive definite matrix A [5].
- Then the HJB equation is equivalent to the **Monge-Ampère** equation:
$$-\frac{1}{2}\det(D^2 v) = f^d, \text{ in } \quad \Omega,$$
with $v = g$ on the boundary, and v convex.
- Equations of this type arise in optimal transport, with right hand side of the form $\frac{f}{g(\nabla v)}$ (see [6]).

Martingale Optimal Transport

- The classical Monge-Kantorovich problem consists of transporting mass from one distribution μ to another ν, minimising a cost c.
- We minimise over probability measures:
$$\inf_{\pi \in \Pi(\mu,\nu)} \int c(x,y)\pi(\mathrm{d}x, \mathrm{d}y),$$
$$\Pi(\mu,\nu) = \left\{\pi : \int \pi(\cdot, \mathrm{d}\nu) = \mu, \int \pi(\mathrm{d}\mu, \cdot) = \nu\right\}.$$



- Martingale optimal transport imposes the additional constraint that, given $X \sim \mu$, then $Y \sim \nu$ has expected value X.
- A Lagrangian formulation of this constrained optimisation problem gives rise to stochastic control problems of the type seen here [7].
- *Fully exploring this connection is future work.*

References

[1] Nizar Touzi.
Optimal stochastic control, stochastic target problems and backward SDE.
Springer-Verlag, 2012.

[2] Wendell H. Fleming and Halil Mete Soner.
Controlled Markov processes and viscosity solutions.
Springer Science & Business Media, 2006.

[3] Bernard Gaveau.
Méthodes de contrôle optimal en analyse complexe. I. Résolution d'équations de Monge Ampère.
Journal of Functional Analysis, 25:391–411, 1977.

[4] Michael G. Crandall, Hitoshi Ishii, and Pierre-Louis Lions.
User's guide to viscosity solutions of second order partial differential equations.
Bulletin of the American Mathematical Society, 27(1):1–67, 1992.

[5] Hitoshi Ishii and Pierre-Luis Lions.
Viscosity solutions of fully nonlinear second-order elliptic partial differential equations.
Journal of Differential equations, 83(1):26–78, 1990.

[6] Cédric Villani.
Topics in optimal transportation.
American Mathematical Soc., 2003.

[7] Alfred Galichon, Pierre Henry-Labordere, and Nizar Touzi.
A stochastic control approach to no-arbitrage bounds given marginals, with an application to lookback options.
The Annals of Applied Probability, 24(1):312–336, 2014.