Bicausal optimal transport for SDEs with irregular coefficients

Benjamin A. Robinson (Universität Klagenfurt)

June 6, 2024

Klagenfurt–Berlin Workshop on Multiple Perspectives in Optimisation

Supported by Austrian Science Fund (FWF) projects (Y782-N25), (P35519), (P34743).

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Adapted Wasserstein distance between the laws of SDEs
(with J. Backhoff-Veraguas and S. Källblad) — arXiv:2209.03243, 2022

Bicausal optimal transport for SDEs with irregular coefficients
(with M. Szölgyenyi) — arXiv:2403.09941, 2024

Comparing stochastic models

Aim: Compute a measure of model uncertainty

E.g.

$$\mathbb{P} \mapsto v(\mathbb{P}) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}}[\mathcal{J}(\omega, \alpha)]$$

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SDEs:

- Good computational methods available
- Rich class of models, beyond Lipschitz coefficients

Main result

$b, \bar{b} \colon [0,T] \times \mathbb{R} \to \mathbb{R}$, $\sigma, \bar{\sigma} \colon [0,T] \times \mathbb{R} \to [0,\infty)$,

$$\begin{aligned} \mathrm{d}X_t &= b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}B_t, \quad X_0 = x_0, \\ \mathrm{d}\bar{X}_t &= \bar{b}_t(\bar{X}_t)\mathrm{d}t + \bar{\sigma}_t(\bar{X}_t)\mathrm{d}W_t, \quad \bar{X}_0 = x_0. \end{aligned}$$

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$$d(\mu,\nu)^p = \mathbb{E}\left[\int_0^T |X_t - \bar{X}_t|^p \mathrm{d}t\right], \quad \text{with } B = W.$$

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$b\colon \mathbb{R} \to \mathbb{R}$, $\sigma\colon \mathbb{R} \to [0,\infty)$, $X_0 = x \in \mathbb{R}$,

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b satisfies piecewise regularity conditions and exponential growth condition,
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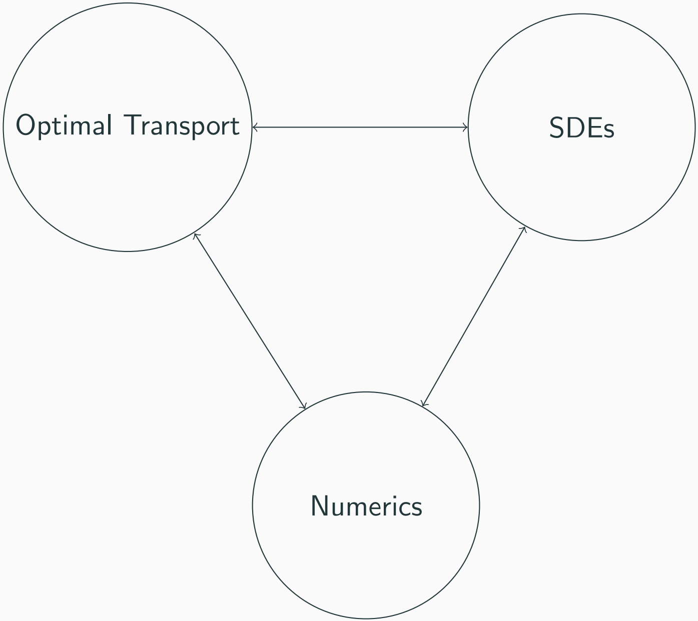
Strong existence, pathwise uniqueness, and moment bounds hold for (SDE) with coefficients satisfying (A). Moreover, for a transformation-based semi-implicit Euler scheme, we obtain strong convergence rates.

Ingredients



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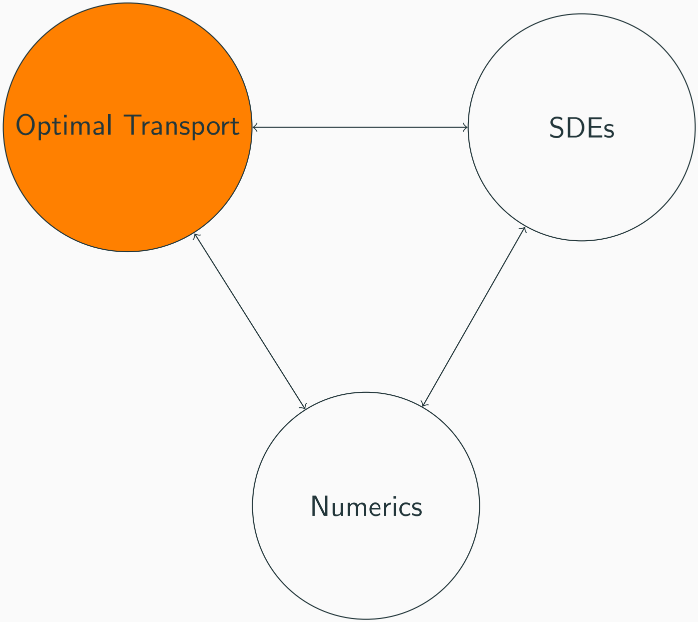

Optimal transport

Probability measures μ, ν on $\mathbb{R}^N$

Find

$$\inf_{T\colon T_\#\mu=\nu} \mathbb{E}\left[\sum_{n=1}^{N} |T_n(X) - X_n|^p\right]$$

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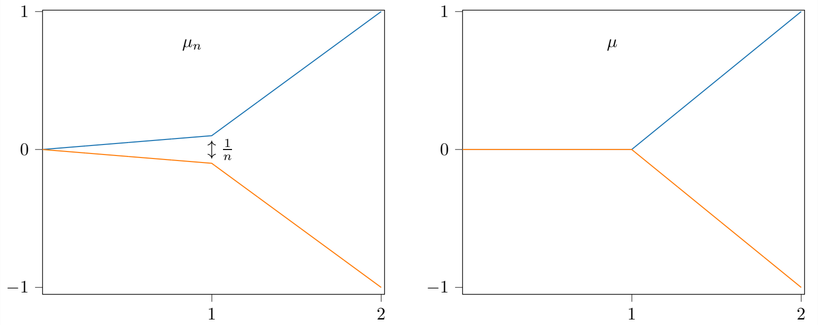
Wasserstein distance metrises usual weak topology

Example

[Backhoff-Veraguas, Bartl, Beiglböck, Eder '20], [Aldous '81]



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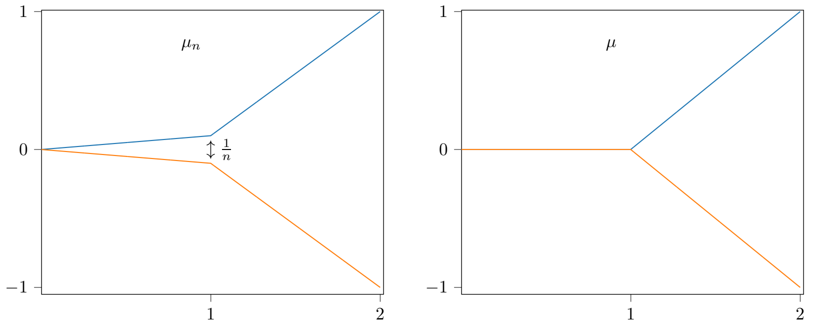

"Can get rich" "Cannot get rich"

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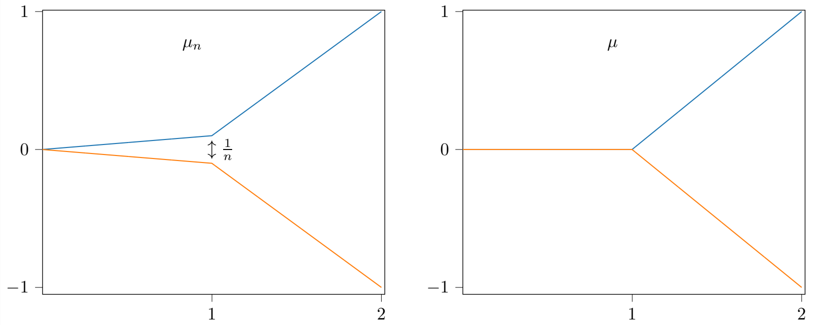

$$V_n := \sup_{\tau} \mathbb{E}^{\mu_n}[X_\tau] \approx \frac{1}{2} \qquad V := \sup_{\tau} \mathbb{E}^{\mu}[X_\tau] = 0$$

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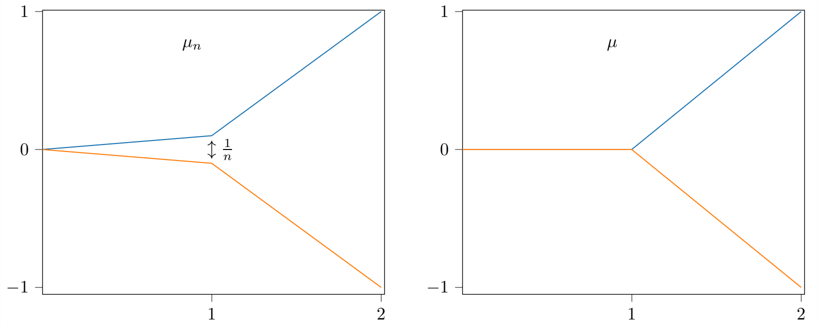

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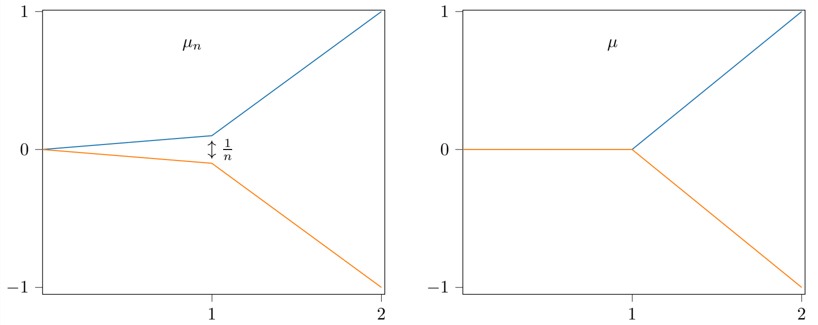

$$V_n \not\to V \quad \text{but} \quad \mu_n \rightharpoonup \mu$$

Adapted topology

$$\mu, \nu \in \mathcal{P}(\mathbb{R}^N) \quad \rightsquigarrow \quad \inf_{T\colon T_\# \mu = \nu} \mathbb{E}\left[\sum_{n=1}^{N} |T_n(X) - X_n|^p\right]$$

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More general cost functions $\rightsquigarrow$ bicausal optimal transport

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c_n continuous, polynomial growth, quasi-monotone

$$c_n(x,y) + c_n(x',y') - c_n(x,y') - c_n(x',y) \geq 0, \quad \forall\, x \leq x', y \leq y'$$

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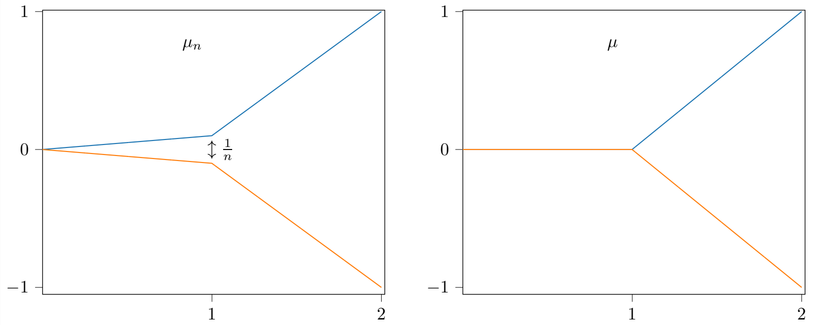
Acciaio, Aldous, Backhoff-Veraguas, Bartl, Beiglböck, Bion-Nadal, Eder, Hellwig, Källblad, Pammer, Pflug, Pichler, Talay, Zalaschko, ...

Example revisited

[Backhoff-Veraguas, Bartl, Beiglböck, Eder '20], [Aldous '81]



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$V_n \not\to V$ and $\mathcal{AW}_p(\mu_n, \mu) \not\to 0$

Continuous time

Similar definition of Wasserstein distance in continuous time w.r.t. L^p norm on $\Omega := C([0,T],\mathbb{R})$

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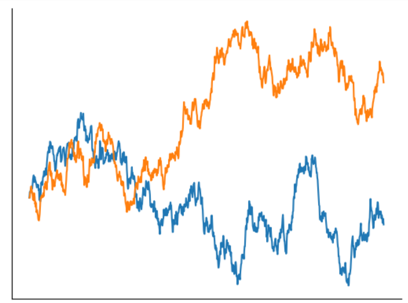

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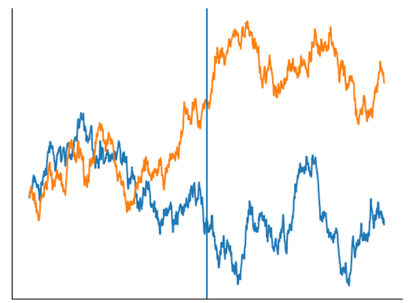

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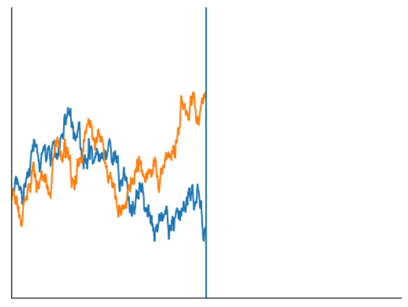

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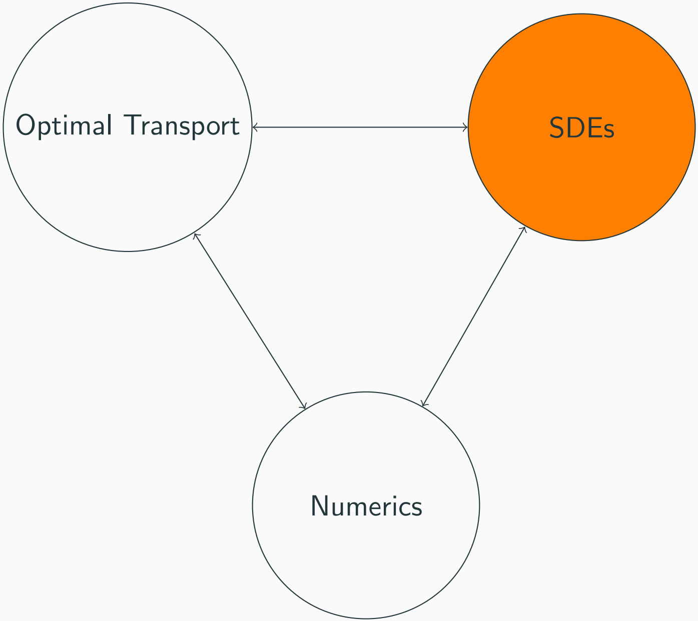

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$$\mathcal{AW}_p^p(\mu,\nu) = \mathbb{E}\Big[\int_0^T |X_t - \bar{X}_t|^p \mathrm{d}t\Big], \quad \text{with } B = W.$$

Coupling SDEs

$$\mathrm{d}X_t = b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}B_t,\ X_0 = x \ \rightsquigarrow\ \mathrm{Law}(X) = \mu$$
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Theorem [Backhoff-Veraguas, Källblad, R. '22]

Optimising over bicausal couplings $\pi \in \mathrm{Cpl}_{\mathrm{bc}}(\mu, \nu)$

$\Leftrightarrow$

Optimising over correlations between B, W

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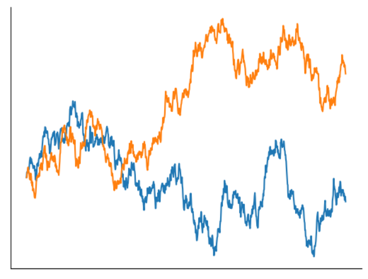
$\Leftrightarrow$
Optimising over correlations between B, W

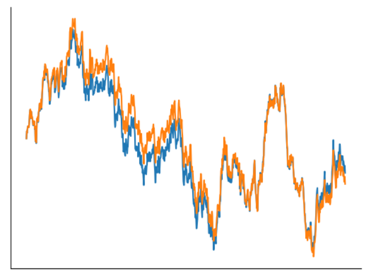
Product coupling

B, W independent



Synchronous coupling

Choose the same driving Brownian motion $B = W$.



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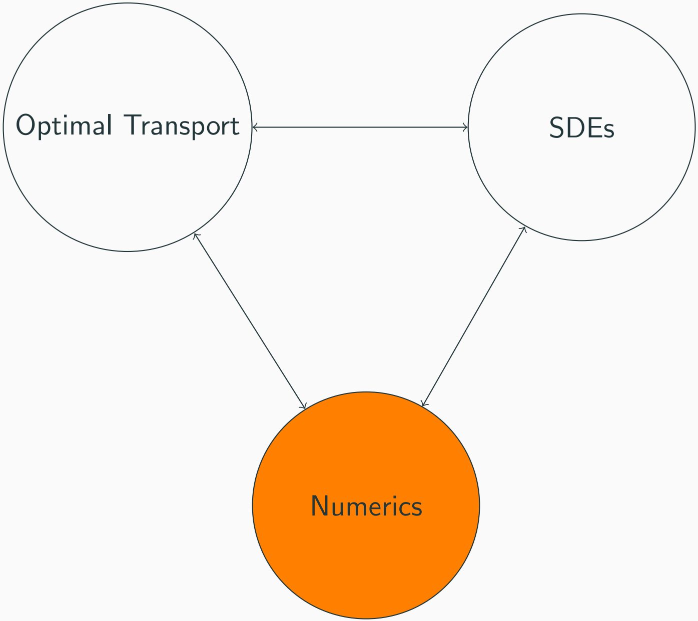

Proof of main result

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Under "weak assumptions" on the coefficients, we can compute the adapted Wasserstein distance by

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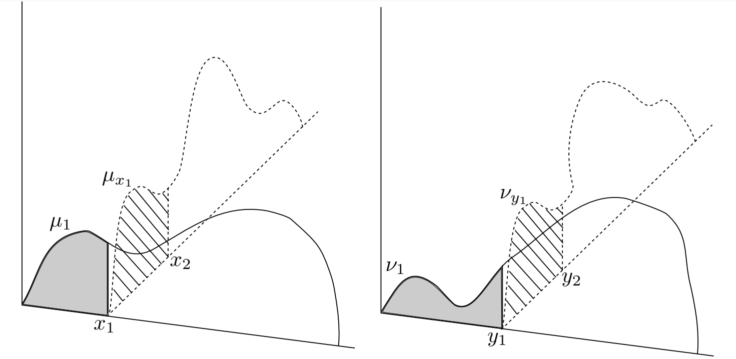
— generalisation of monotone rearrangement



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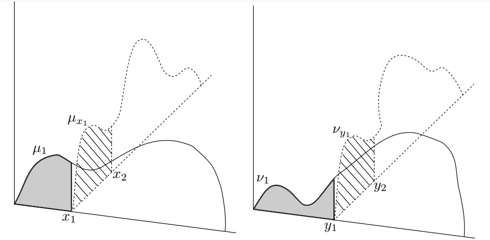

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Theorem [Rüschendorf '85] [Posch '23]

For μ, ν stochastically co-monotone, the unique optimiser is the Knothe–Rosenblatt rearrangement.

This induces the adapted weak topology.

Proof of main result

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A monotone numerical scheme

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t$$

Euler scheme

$$X_0^h = X_0,$$
$$X_t^h = X_{kh}^h + b(X_{kh})(t - kh), \quad t \in (kh, (k+1)h].$$

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Lemma [Backhoff-Veraguas, Källblad, R. '22]

For b, σ Lipschitz, the monotone Euler–Maruyama scheme is stochastically increasing.

Hence the Knothe–Rosenblatt rearrangement is optimal for μ^h, ν^h.

Proof of main result

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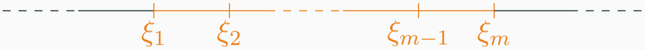
Transformation-based semi-implicit Euler scheme

Assumption (A)

Drift $b\colon \mathbb{R} \to \mathbb{R}$ satisfies the following conditions piecewise:

- absolute continuity
- one-sided Lipschitz condition
- two-sided local Lipschitz condition
- exponential growth



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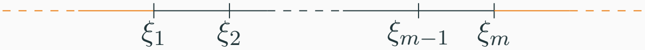

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- global Lipschitz condition
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Under Assumption (A) , the scheme is constructed as follows:

1. Apply the transformation G from [Leobacher, Szölgyenyi '17] to (SDE),

$$Z = G(X)$$

$$\mathrm{d}Z_t = \tilde{b}(Z_t)\mathrm{d}t + \tilde{\sigma}(Z_t)\mathrm{d}W_t$$

$\tilde{b}$ one-sided Lipschitz, exponential growth, locally Lipschitz, a.c.

$\tilde{\sigma}$ Lipschitz

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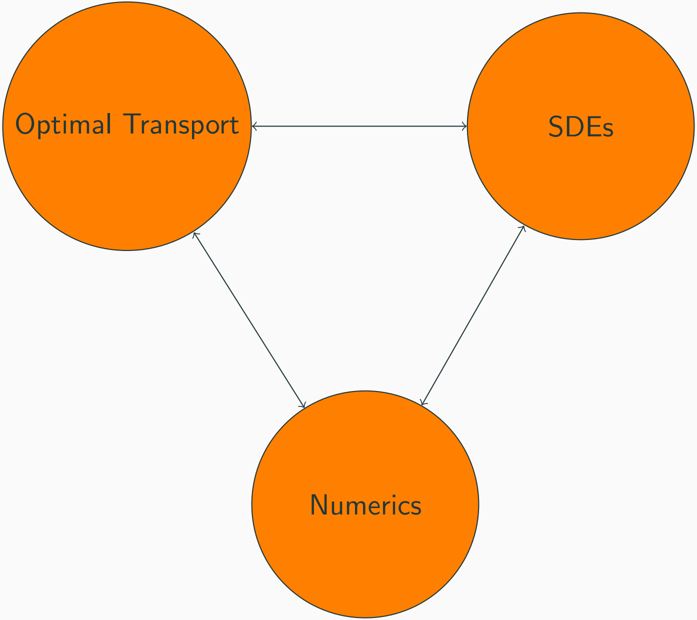
Theorem [R., Szölgyenyi '24]

Let (b, σ) satisfy Assumption (A). Then (SDE) admits a unique strong solution and, for all $p \geq 1$, there exists $C_p \geq 0$ such that

$$\mathbb{E}\Big[|X_T - X_T^h|^p\Big]^{\frac{1}{p}} \leq \begin{cases} C_p h^{\frac{1}{2}}, & p \in [1,2], \\ C_p h^{\frac{1}{p(p-1)}}, & p \geq 2. \end{cases}$$

Ingredients



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Synchronous coupling solves general bicausal transport problem

Future research directions

- Extension to higher dimensions
  - Examples in [Backhoff-Veraguas, Källblad, R. '22] show that the synchronous coupling is not always optimal
- Extension to jump-diffusions
- Extension to neural SDEs, McKean–Vlasov SDEs
- Convergence of optimisers
  - Use density estimates for SDEs from [Backhoff-Veraguas, Unterberger '23]
- Application to uniqueness of mimicking martingales

Summary

- We compute adapted Wasserstein distance between SDEs with irregular coefficients
- We prove strong convergence rates for a numerical scheme for SDEs with discontinuous and exponentially growing drift

References: