Bicausal optimal transport for SDEs with irregular coefficients

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Joint work with

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Sigrid Källblad
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Adapted Wasserstein distance between the laws of SDEs
(with J. Backhoff-Veraguas and S. Källblad) — `http://arxiv.org/abs/2209.03243`

Bicausal optimal transport for SDEs with irregular coefficients
(with M. Szölgyenyi) — `https://arxiv.org/abs/2403.09941`

Comparing stochastic models

Aim: Compute a measure of model uncertainty

E.g.

$$\mathbb{P} \mapsto v(\mathbb{P}) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}}[\mathcal{J}(\omega, \alpha)]$$

Want:

- Appropriate topology on laws of stochastic processes
- Distance we can actually compute

SDEs:

- Good computational methods available
- Rich class of models, beyond Lipschitz coefficients

Main result

$b, \bar{b}\colon [0,T] \times \mathbb{R} \to \mathbb{R}$, $\sigma, \bar{\sigma}\colon [0,T] \times \mathbb{R} \to [0,\infty)$,

$$\mathrm{d}X_t = b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}B_t, \quad X_0 = x_0,$$
$$\mathrm{d}\bar{X}_t = \bar{b}_t(\bar{X}_t)\mathrm{d}t + \bar{\sigma}_t(\bar{X}_t)\mathrm{d}W_t, \quad \bar{X}_0 = x_0.$$

$\mu = \mathrm{Law}(X)$, $\nu = \mathrm{Law}(\bar{X})$

Theorem [R., Szölgyenyi '24+]

Under “weak assumptions” on the coefficients, we can compute an “appropriate distance” by

$$d(\mu,\nu)^p = \mathbb{E}\Big[\int_0^T |X_t - \bar{X}_t|^p \mathrm{d}t\Big], \quad \text{with } B = W.$$

Auxiliary result

$b\colon \mathbb{R} \to \mathbb{R}$, $\sigma\colon \mathbb{R} \to [0,\infty)$, $X_0 = x \in \mathbb{R}$,

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t. \tag{SDE}$$

Assumption (A)

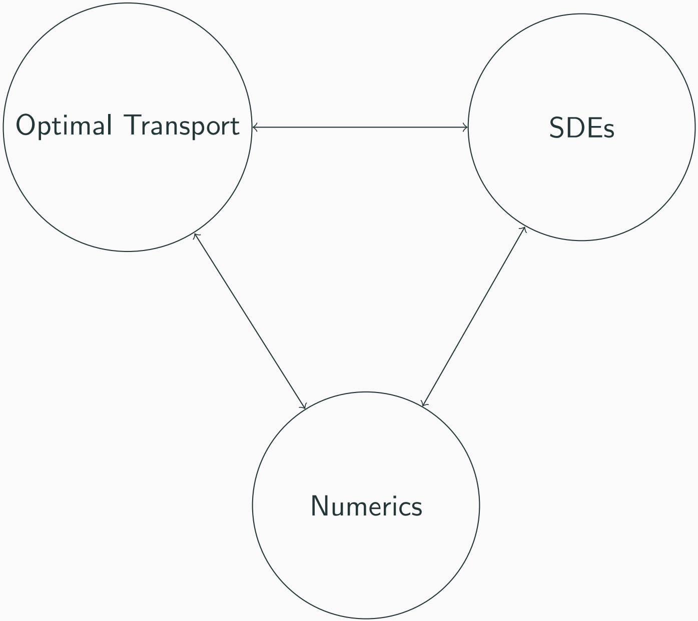
b satisfies piecewise regularity conditions and exponential growth condition,

σ is Lipschitz and non-zero at the discontinuity points of b.

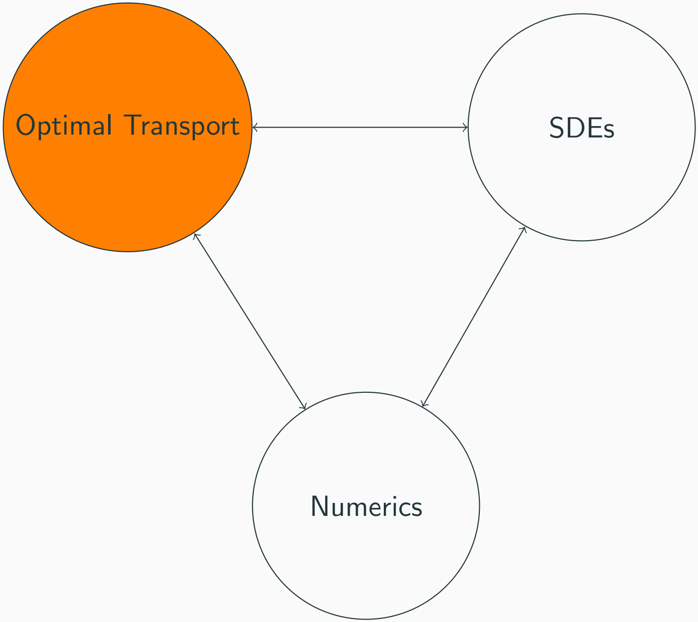
Theorem [R., Szölgyenyi '24+]

Strong existence, pathwise uniqueness, and moment bounds hold for (SDE) with coefficients satisfying (A). Moreover, for a transformation-based semi-implicit Euler scheme, we obtain strong convergence rates.

Ingredients



Ingredients



Optimal transport

Probability measures μ, ν on $\mathbb{R}^N$

Find

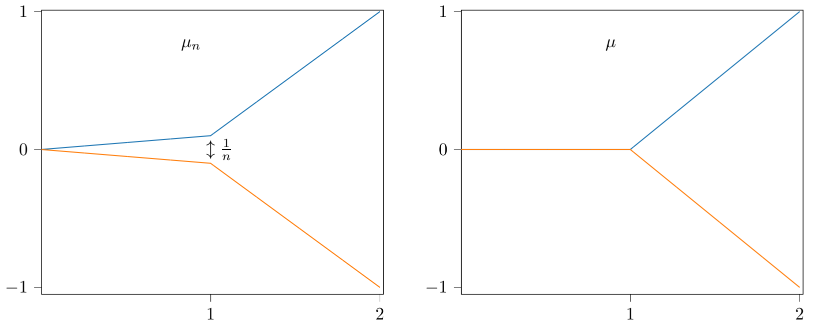
$$\mathcal{W}_2^2(\mu,\nu) := \inf_{T:\, T_\#\mu=\nu} \mathbb{E}\left[\sum_{n=1}^{N} |T_n(X) - X_n|^2\right].$$

$$T(X) = (T_1(X_1,\ldots,X_N),\ldots,T_N(X_1,\ldots,X_N))$$

Monge (1781)
Kantorovich (1942) $\rightsquigarrow$ T random: replace $(X, T(X))$ with coupling (X,Y), $X \sim \mu$, $Y \sim \nu$
Wasserstein distance metrises usual weak topology

Example

[Backhoff-Veraguas, Bartl, Beiglböck, Eder '20], [Aldous '81]



"Can get rich" "Cannot get rich"

$$V_n := \sup_\tau \mathbb{E}^{\mu_n}[X_\tau] \approx \frac{1}{2} \qquad V := \sup_\tau \mathbb{E}^{\mu}[X_\tau] = 0$$

$V_n \not\to V$ but $\mu_n \to \mu$

Adapted topology

$$\mu, \nu \in \mathcal{P}(\mathbb{R}^N) \quad \rightsquigarrow \quad \inf_{T\colon T_\# \mu = \nu} \mathbb{E}\left[\sum_{n=1}^{N} |T_n(X) - X_n|^p\right]$$

$$T(X) = (T_1(X_1, \ldots, X_N), \ldots, T_N(X_1, \ldots, X_N))$$

Adapted topology

$$\mu, \nu \in \mathcal{P}(\mathbb{R}^N) \quad \rightsquigarrow \quad \inf_{\substack{T\colon T_\# \mu = \nu \\ \textit{adapted}}} \mathbb{E}\left[\sum_{n=1}^{N} |T_n(X) - X_n|^p\right]$$

$$T(X) = (T_1(X_1), T_2(X_1, X_2), \ldots, T_N(X_1, \ldots, X_N))$$

Adapted topology

$$\mu, \nu \in \mathcal{P}(\mathbb{R}^N) \quad \rightsquigarrow \quad \mathcal{AW}_p^p(\mu, \nu) := \inf_{\pi \in \mathrm{Cpl}_{\mathrm{bc}}(\mu,\nu)} \mathbb{E}^\pi \left[\sum_{n=1}^N |X_n - Y_n|^p \right]$$

$$\mathrm{Cpl}_{\mathrm{bc}}(\mu, \nu) = \{\pi \in \mathrm{Cpl}(\mu, \nu) \colon \ \pi \text{ bicausal}\}$$

Adapted topology

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More general cost functions $\rightsquigarrow$ bicausal optimal transport

$$\inf_{\pi \in \mathrm{Cpl}_{\mathrm{bc}}(\mu,\nu)} \mathbb{E}^\pi \left[\sum_{n=1}^N c_n(X_n, Y_n) \right]$$

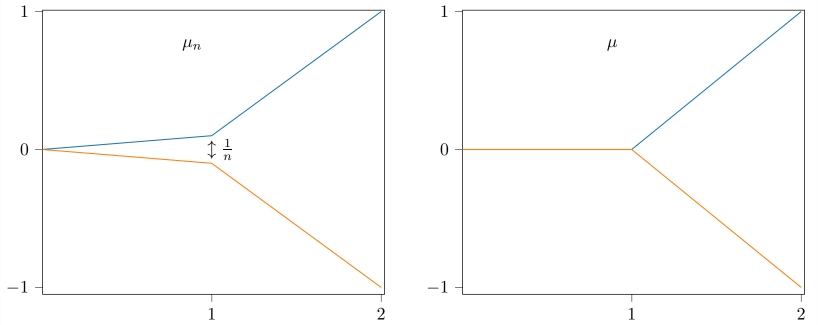
c_n continuous, polynomial growth, quasi-monotone

$$c_n(x, y) + c_n(x', y') - c_n(x, y') - c_n(x', y) \geq 0, \quad \forall\, x \leq x', y \leq y'$$

Acciaio, Aldous, Backhoff-Veraguas, Bartl, Beiglböck, Bion-Nadal, Eder, Hellwig, Källblad, Pammer, Pflug, Pichler, Talay, Zalaschko, ...

Example revisited

[Backhoff-Veraguas, Bartl, Beiglböck, Eder '20], [Aldous '81]



$$V_n := \sup_\tau \mathbb{E}^{\mu_n}[X_\tau] \approx \frac{1}{2} \qquad V := \sup_\tau \mathbb{E}^{\mu}[X_\tau] = 0$$

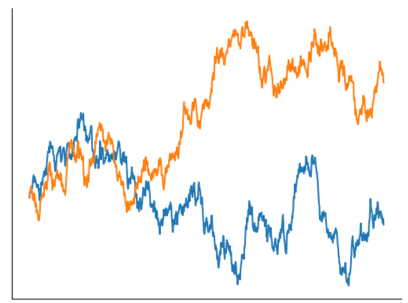
$V_n \not\to V$ and $\mathcal{AW}_p(\mu_n, \mu) \not\to 0$

Continuous time

Similar definition of Wasserstein distance in continuous time w.r.t. L^p norm on $\Omega := C([0,T],\mathbb{R})$

$$\mu,\nu \in \mathcal{P}(\Omega) \quad \rightsquigarrow \quad \mathcal{W}_p^p(\mu,\nu) := \inf_{\pi\in \mathrm{Cpl}(\mu,\nu)} \mathbb{E}^\pi\left[\int_0^T |\omega_t - \bar{\omega}_t|^p \mathrm{d}t\right]$$

$$\mathrm{Cpl}(\mu,\nu) = \{\pi = \mathrm{Law}(X,Y)\colon\ X\sim\mu, Y\sim\nu\}$$



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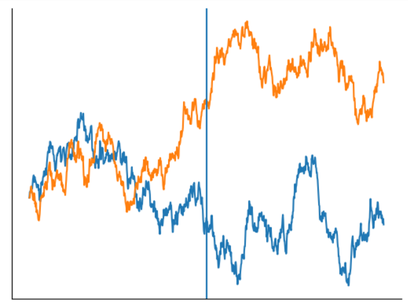
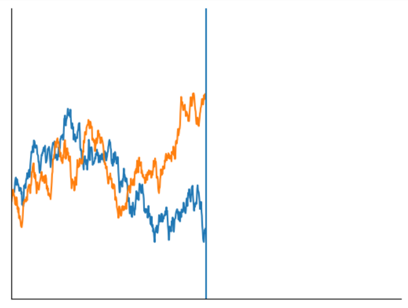


Continuous time

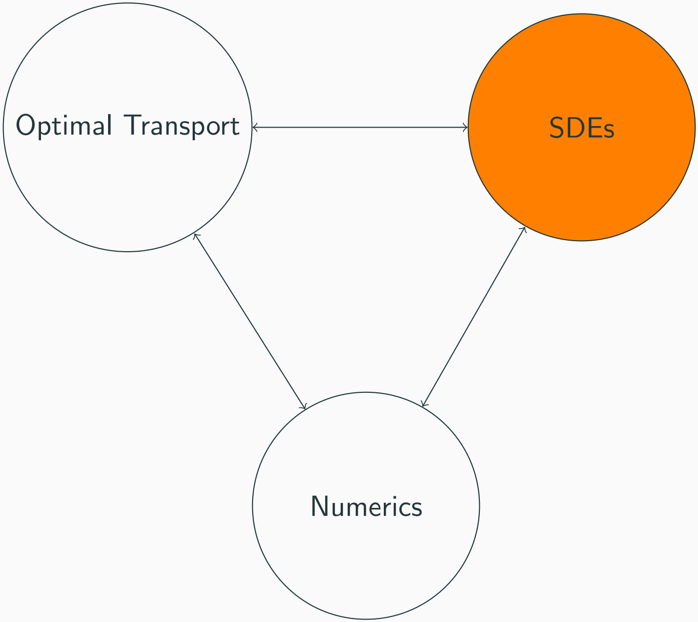
Similar definition of adapted Wasserstein distance in continuous time w.r.t. L^p norm on $\Omega := C([0,T],\mathbb{R})$

$$\mu,\nu \in \mathcal{P}(\Omega) \quad \rightsquigarrow \quad \mathcal{AW}_p^p(\mu,\nu) := \inf_{\pi \in \mathrm{Cpl}_{\mathrm{bc}}(\mu,\nu)} \mathbb{E}^\pi \left[\int_0^T |\omega_t - \bar{\omega}_t|^p \mathrm{d}t \right]$$

$$\mathrm{Cpl}_{\mathrm{bc}}(\mu,\nu) = \{\pi \in \mathrm{Cpl}(\mu,\nu) \colon \ \pi \text{ bicausal}\}$$



Ingredients



Main result

$b, \bar{b} \colon [0,T] \times \mathbb{R} \to \mathbb{R}$, $\sigma, \bar{\sigma} \colon [0,T] \times \mathbb{R} \to [0,\infty)$,

$$\begin{aligned} \mathrm{d}X_t &= b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}B_t, \quad X_0 = x_0, \\ \mathrm{d}\bar{X}_t &= \bar{b}_t(\bar{X}_t)\mathrm{d}t + \bar{\sigma}_t(\bar{X}_t)\mathrm{d}W_t, \quad \bar{X}_0 = x_0. \end{aligned}$$

$\mu = \mathrm{Law}(X)$, $\nu = \mathrm{Law}(\bar{X})$

Theorem [R., Szölgyenyi '24+]

Under "weak assumptions" on the coefficients, we can compute the adapted Wasserstein distance by

$$\mathcal{AW}_p^p(\mu,\nu) = \mathbb{E}\Big[\int_0^T |X_t - \bar{X}_t|^p \mathrm{d}t\Big], \quad \text{with } B = W.$$

Coupling SDEs

$$\mathrm{d}X_t = b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}B_t,\ X_0 = x \ \rightsquigarrow\ \mathrm{Law}(X) = \mu$$
$$\mathrm{d}\bar{X}_t = \bar{b}_t(\bar{X}_t)\mathrm{d}t + \bar{\sigma}_t(\bar{X}_t)\mathrm{d}W_t,\ \bar{X}_0 = x \ \rightsquigarrow\ \mathrm{Law}(\bar{X}) = \nu$$

Theorem [Backhoff-Veraguas, Källblad, R. '22]

Optimising over bicausal couplings $\pi \in \mathrm{Cpl}_{\mathrm{bc}}(\mu, \nu)$

$\Leftrightarrow$

Optimising over correlations between B, W

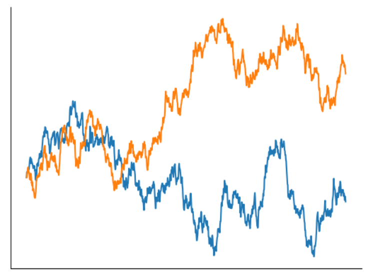
Coupling SDEs

Theorem [Backhoff-Veraguas, Källblad, R. '22]

$$\text{Optimising over bicausal couplings } \pi \in \mathrm{Cpl}_{\mathrm{bc}}(\mu,\nu)$$
$$\Leftrightarrow$$
$$\text{Optimising over correlations between } B, W$$

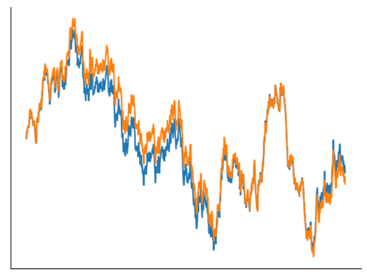
Product coupling

B, W independent



Synchronous coupling

Choose the same driving Brownian motion $B = W$.



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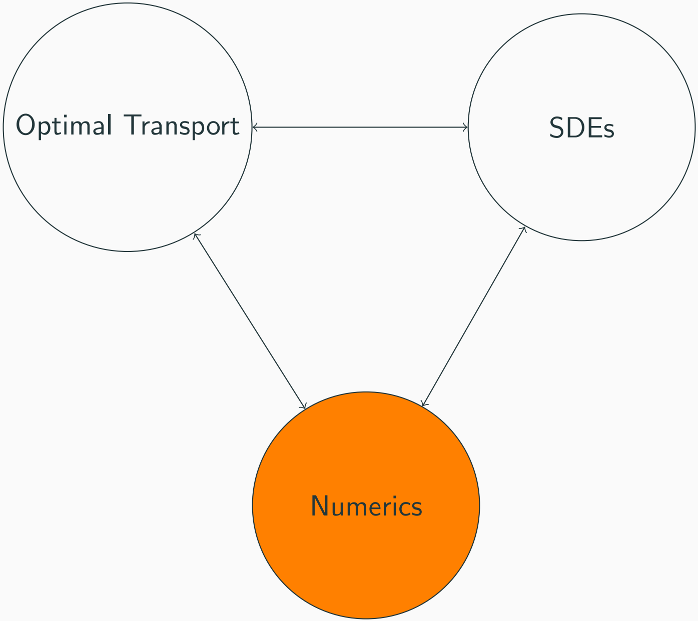

Proof of main result

$$\mathrm{d}X_t = b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}B_t,\ X_0 = x \ \rightsquigarrow \ \mathrm{Law}(X) = \mu$$
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Theorem [R., Szölgyenyi '24+]

Under "weak assumptions" on the coefficients, we can compute the adapted Wasserstein distance by

$$\mathcal{AW}_p^p(\mu,\nu) = \mathbb{E}\left[\int_0^T |X_t - \bar{X}_t|^p \mathrm{d}t\right], \quad \text{with } B = W.$$

1. Discretise SDEs;
2. Solve discrete-time bicausal optimal transport problem;
3. Pass to a limit.

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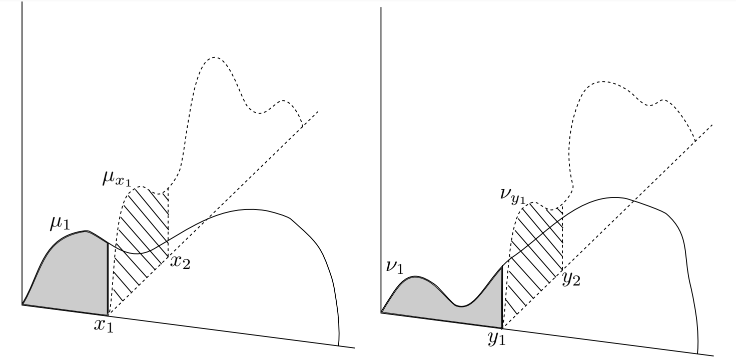
Key result in discrete time

$$\mu, \nu \in \mathcal{P}(\mathbb{R}^N) \quad \rightsquigarrow \quad \mathcal{AW}_p^p(\mu, \nu) := \inf_{\pi \in \mathrm{Cpl}_{\mathrm{bc}}(\mu,\nu)} \mathbb{E}^\pi \left[\sum_{n=1}^N |X_n - Y_n|^p \right]$$

Knothe–Rosenblatt rearrangement

— generalisation of monotone rearrangement



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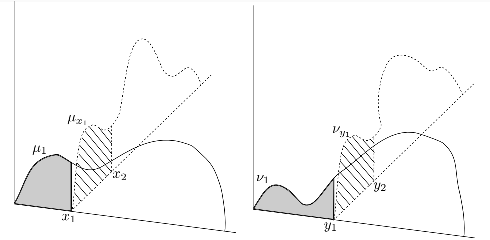
Knothe–Rosenblatt rearrangement

$$Y_k = T_k^{\mathrm{KR}}(X_1, \ldots, X_k) = F^{-1}_{\nu_{Y_1,\ldots,Y_{k-1}}} \circ F_{\mu_{X_1,\ldots,X_{k-1}}}(X_k),$$



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Knothe–Rosenblatt rearrangement

$$Y_k = T_k^{\mathrm{KR}}(X_1, \ldots, X_k) = F^{-1}_{\nu_{Y_1,\ldots,Y_{k-1}}} \circ F_{\mu_{X_1,\ldots,X_{k-1}}}(X_k),$$

Theorem [Rüschendorf ’85] [Posch ’23+]

For μ, ν stochastically co-monotone, the unique optimiser is the Knothe–Rosenblatt rearrangement.

This induces the adapted weak topology.

Proof of main result

$$\mathrm{d}X_t = b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}B_t,\ X_0 = x \ \rightsquigarrow\ \mathrm{Law}(X) = \mu$$
$$\mathrm{d}\bar{X}_t = \bar{b}_t(\bar{X}_t)\mathrm{d}t + \bar{\sigma}_t(\bar{X}_t)\mathrm{d}W_t,\ \bar{X}_0 = x \ \rightsquigarrow\ \mathrm{Law}(\bar{X}) = \nu$$

Theorem [R., Szölgyenyi '24+]

Under "weak assumptions" on the coefficients, we can compute the adapted Wasserstein distance by

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1. Discretise SDEs;
2. Solve discrete-time bicausal optimal transport problem;
3. Pass to a limit.

A monotone numerical scheme

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t$$

Euler scheme

$$X_0^h = X_0,$$
$$X_t^h = X_{kh}^h + b(X_{kh})(t - kh), \quad t \in (kh, (k+1)h].$$

A monotone numerical scheme

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}W_t$$

Euler–Maruyama scheme

$$X_0^h = X_0,$$

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Write $X_k^h := X_{kh}^h$ and $\mu^h = \mathrm{Law}((X_k^h)_k)$.

Remark

$X_k^h \mapsto X_{(k+1)}^h$ is increasing if b is Lipschitz, $h \ll 1$

A monotone numerical scheme

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t$$

Monotone Euler–Maruyama scheme

$X_0^h = X_0,$

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$W_t^h - W_{kh}^h = W_{t\wedge\tau_k^h} - W_{kh}, \quad \tau_k^h := \inf\{t > kh\colon\ |W_t - W_{kh}| > A_h|\}$

A monotone numerical scheme

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Monotone Euler–Maruyama scheme

$X_0^h = X_0,$

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Write $X_k^h := X_{kh}^h$ and $\mu^h = \mathrm{Law}((X_k^h)_k)$.

Lemma [Backhoff-Veraguas, Källblad, R. '22]

For b, σ Lipschitz, the monotone Euler–Maruyama scheme is stochastically increasing.

Hence the Knothe–Rosenblatt rearrangement is optimal for μ^h, ν^h.

Proof of main result

$$\mathrm{d}X_t = b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}B_t,\ X_0 = x \ \rightsquigarrow\ \mathrm{Law}(X) = \mu$$
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Theorem [R., Szölgyenyi ’24+]

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Transformation-based semi-implicit Euler scheme

Assumption (A)

Drift $b\colon \mathbb{R} \to \mathbb{R}$ satisfies the following conditions piecewise:

- absolute continuity
- one-sided Lipschitz condition
- two-sided local Lipschitz condition
- exponential growth

Diffusion $\sigma\colon \mathbb{R} \to [0, \infty)$ satisfies

- global Lipschitz condition
- $\sigma(\xi_k) \neq 0$, for $k \in \{1, \ldots, m\}$ — no uniform ellipticity

Transformation-based semi-implicit Euler scheme

Under Assumption (A) , the scheme is constructed as follows:

1. Apply the transformation G from [Leobacher, Szölgyenyi '17] to (SDE),

$$Z = G(X)$$

$$\mathrm{d}Z_t = \tilde{b}(Z_t)\mathrm{d}t + \tilde{\sigma}(Z_t)\mathrm{d}W_t$$

$\tilde{b}$ one-sided Lipschitz, exponential growth, locally Lipschitz, a.c.

$\tilde{\sigma}$ Lipschitz

Transformation-based semi-implicit Euler scheme

Under Assumption (A) , the scheme is constructed as follows:

1. Apply the transformation G from [Leobacher, Szölgyenyi '17] to (SDE),
2. Apply a semi-implicit Euler scheme with truncated Brownian increments to the transformed SDE,

$$Z = G(X)$$

$$\mathrm{d}Z_t = \tilde{b}(Z_t)\mathrm{d}t + \tilde{\sigma}(Z_t)\mathrm{d}W_t$$

$$Z^h_{(k+1)h} = Z^h_{kh} + \tilde{b}(Z^h_{(k+1)h}) \cdot h + \tilde{\sigma}(Z^h_{kh})(W^h_{(k+1)h} - W^h_{kh})$$

Transformation-based semi-implicit Euler scheme

Under Assumption (A) , the scheme is constructed as follows:

1. Apply the transformation G from [Leobacher, Szölgyenyi '17] to (SDE),
2. Apply a semi-implicit Euler scheme with truncated Brownian increments to the transformed SDE,
3. Transform back.

$$Z = G(X)$$

$$\mathrm{d}Z_t = \tilde{b}(Z_t)\mathrm{d}t + \tilde{\sigma}(Z_t)\mathrm{d}W_t$$

$$Z^h_{(k+1)h} = Z^h_{kh} + \tilde{b}(Z^h_{(k+1)h}) \cdot h + \tilde{\sigma}(Z^h_{kh})(W^h_{(k+1)h} - W^h_{kh})$$

$$X^h_{kh} = G^{-1}(Z^h_{kh})$$

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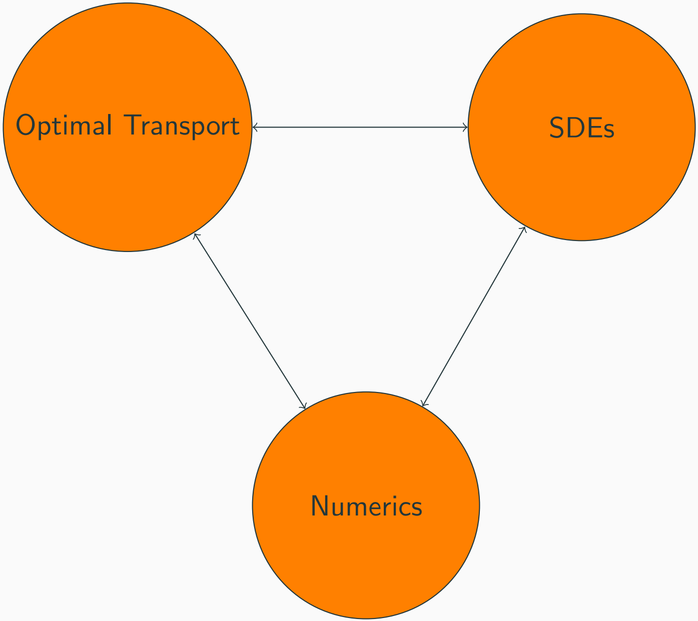
3. Transform back.

Theorem [R., Szölgyenyi '24]

Let (b, σ) satisfy Assumption (A). Then (SDE) admits a unique strong solution and, for all $p \geq 1$, there exists $C_p \geq 0$ such that

$$\mathbb{E}\Big[|X_T - X_T^h|^p\Big]^{\frac{1}{p}} \leq \begin{cases} C_p h^{\frac{1}{2}}, & p \in [1,2], \\ C_p h^{\frac{1}{p(p-1)}}, & p \geq 2. \end{cases}$$

Ingredients



Main result

Assumptions
(A) discontinuous drift with exponential growth (time-homog.);
(B) bounded measurable drift, α-Hölder and uniformly elliptic σ;
(C) continuous coefficients, linear growth, pathwise uniqueness.

$$\mathrm{d}X_t = b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}B_t,\ X_0 = x \ \rightsquigarrow\ \mathrm{Law}(X) = \mu$$
$$\mathrm{d}\bar{X}_t = \bar{b}_t(\bar{X}_t)\mathrm{d}t + \bar{\sigma}_t(\bar{X}_t)\mathrm{d}W_t,\ \bar{X}_0 = x \ \rightsquigarrow\ \mathrm{Law}(\bar{X}) = \nu$$

Main Theorem [R., Szölgyenyi '24+]

Let (b, σ) and $(\bar{b}, \bar{\sigma})$ each satisfy one of assumptions (A), (B), (C). Then, for $p \in [1, \infty)$, the adapted Wasserstein distance is given by

$$\mathcal{AW}_p^p(\mu, \nu) = \mathbb{E}\Big[\int_0^T |X_t - \bar{X}_t|^p \mathrm{d}t\Big], \text{ with } B = W$$

Synchronous coupling solves general bicausal transport problem

Future research directions

- Extension to higher dimensions
  - Examples in [Backhoff-Veraguas, Källblad, R. '22] show that the synchronous coupling is not always optimal
- Extension to jump-diffusions
- Extension to neural SDEs, McKean–Vlasov SDEs
- Convergence of optimisers
  - Use density estimates for SDEs from [Backhoff-Veraguas, Unterberger '23]
- Application to uniqueness of mimicking martingales

Summary

- We compute adapted Wasserstein distance between SDEs with irregular coefficients
- We prove strong convergence rates for a numerical scheme for SDEs with discontinuous and exponentially growing drift

References: