universität wien

MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

„The Skorokhod Embedding Problem and its Financial Applications"

verfasst von / submitted by

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angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2024 / Vienna, 2024

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	UA 066821
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Mathematics
Betreut von / Supervisor:	Ass.-Prof. Dr. Julio Backhoff-Veraguas
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Danksagung

Ich widme diese Arbeit meinen Eltern, die mich nicht nur in meiner gesamten Bildungskarriere, sondern auch darüber hinaus stets unterstützt haben. Mein Dank gilt meinen ehemaligen Mathematiklehrern Hannes Nägerl und Daniela Hasitzka, die stets bemüht waren und einen bedeutenden Anteil daran haben, dass ich mich für das Studium der Mathematik entschieden habe. Ich bedanke mich bei meinem guten Freund Frederick Franz Moscatelli, der mich bei einigen mathematischen Fragen und Unklarheiten, die im Zuge dieser Arbeit aufgetaucht sind, beraten und mir beim Korrekturlesen geholfen hat. Schlussendlich bedanke ich mich bei meinen Betreuern Benjamin Robinson und Julio Backhoff-Veraguas, die mir dieses Thema empfohlen, mir regelmäßig bei wesentlichen Fragen geholfen und mir Literatur zur Verfügung gestellt haben.

Ralf Stoiber, am 18. Februar 2024

Abriss

Das Ziel dieser Arbeit ist es, das *Skorokhod'sche Einbettungsproblem* und seine Lösungsmethoden zu nutzen, um Theoreme im Feld der Finanzmathematik zu beweisen. Wir sind vorwiegend daran interessiert, Preisuntergrenzen für gewisse Finanzoptionen zu ermitteln. Hierbei wollen wir allerdings kein konkretes stochastisches Modell verwenden.

In Kapitel 1 stellen wir grundlegende Theorie zur *Brown'schen Zufallsbewegung*, zu *Martingalen* und zum *Stochastischen Integral* vor, um sicherzustellen, dass diese Arbeit von allen Studierenden mit einem Basiswissen über Maßtheorie und Wahrscheinlichkeitstheorie gelesen werden kann. Wir formulieren Sätze wie das *Optional Sampling Theorem*, *Itô's Formel* oder *Radon–Nikodym* und zeigen anhand von Beispielen, wie diese Sätze verwendet werden können. Weiters stellen wir grundlegende Begriffe, Resultate und Konzepte aus dem Feld der Finanzmathematik vor. Insbesondere definieren und erklären wir *Optionen*, das Konzept der Preisfindung mittels äquivalenten Martingalmaßen, das *Black–Scholes Modell*, *replizierende Handelsstrategien* und *Arbitrage*.

Im zweiten Kapitel zeigen wir anhand eines motivierenden Beispieles, wie man eine modellunabhängige Preisuntergrenze für die sogenannte *digital option* finden kann. Weiters formulieren und beweisen wir eine Version des Satzes von *Breeden–Litzenberger* und stellen das Konzept von *candidate price processes* vor.

Wir stellen im dritten Kapitel das *Skorokhod'sche Einbettungsproblem* vor und diskutieren die Lösungen von *Doob*, *Hall* und *Root*. Wir gehen der Frage nach Existenz und Eigenschaften von Lösungen nach und zeigen eine Optimalitätseigenschaft von Root's Lösung.

Schlussendlich widmen wir uns in Kapitel 4 den finanzmathematischen Anwendungen des Einbettungsproblems. Wir verwenden Root's Lösung und dessen Optimalität, wobei wir zwei verschiedenen Ansätzen folgen: Zunächst definieren wir das sogenannte *Root Model*, um das Verhalten der zugrundeliegenden Aktie zu beschreiben, und zeigen, dass der in diesem Modell berechnete Preis minimal ist und somit als untere Schranke dient. Im zweiten Ansatz verwenden wir die Konstruktionen vom Beweis zu Root's Optimalität, um ein *subreplizierendes Portfolio* für die gegebene Option zu ermitteln. Der Preis dieses Portfolios liefert uns ebenfalls eine modellunabhängige Preisuntergrenze.

Abstract

The goal of this thesis is to use the *Skorokhod Embedding Problem* and the methods to solve it to prove statements in the field of mathematical finance. Mainly we are interested in finding lower bounds for prices of options on volatility without specifying a concrete stochastic model.

In chapter 1 we introduce some basic theory about *Brownian motion*, *Martingales* and *Stochastic Integral* to make sure this thesis can be read by anyone with fundamental knowledge about measure theory and probability theory. We state theorems like the *Optional Sampling Theorem*, *Itô's Formula* and *Radon–Nikodym* and show with the help of examples how they can be used. Further, we provide the reader with a detailed introduction of important terms, theorems and concepts from the field of mathematical finance. In particular, we will define and explain *options*, the concept of *risk-neutral pricing*, the *Black–Scholes Model*, *hedging strategies*, *arbitrage*.

In chapter 2 we give a motivating example of a *model-independent bound* for the price of a *digital option*. We formulate and prove a version of the theorem of *Breeden and Litzenberger*, and introduce the concept of *candidate price processes*.

We introduce the *Skorokhod Embedding Problem* in Chapter 3 and discuss solutions of *Doob*, *Hall* and *Root*. We answer the question about existence of solutions and establish basic properties of such. Further, we explain in which sense Root's solution is optimal and provide the reader with a detailed proof of this optimality property.

Last but not least, in chapter 4 we establish bounds for prices of options based on volatility. An important tool is the Root solution of the Skorokhod Embedding Problem. We will follow two different approaches: First, we are going to define the *Root Model* to describe the behavior of the underlying stock, and show that prices computed by any other model cannot be below the price of the option in the Root model by the optimality property. In the second approach, we use constructions from the proof of optimality to determine a subhedging portfolio for the option. The amount of money necessary to acquire this portfolio yields a bound for the price of the option.

1 Basic Concepts and Terminology

In this section we follow mostly (but not only) the books of Lamberton [9], Le Gall [10], Shreve [17] and Karatzas & Shreve [8]. We will give a brief overview of basic definitions and properites of Brownian motion and martingales, and recall some basic properties. We will define variation and many forms of the stochastic integral. Further, we'll discuss important results from stochastic analysis such as Itô's formula or Girsanov. We'll introduce the Black–Scholes Model and hedging strategies, and the concept of pricing options. Despite this introduction of the field of stochastic analysis and mathematical finance, some knowledge about measure theory, probability theory and stochastic processes is required.

In this chapter we will always assume that all processes are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

1.1 Brownian Motion

Definition 1.1.
We call a real valued stochastic process $W = (W_t)_{t\geq 0}$ a **Brownian motion**, if...

i) ...it is continuous, i.e. if the map $s \mapsto W_s(\omega)$ is continuous $\mathbb{P}$-a.s.

ii) ...the increments are independent, i.e for all $s \leq t$ we have that $W_t - W_s$ is independent of $\mathcal{F}_s$

iii) ...the increments are stationary, i.e for all $s \leq t$ we have that $W_t - W_s$ and $W_{t-s} - W_0$ have the same law.

We call W a **standard Brownian motion**, if W satisfies i)-iii) and we additionally have:

$$W_0 = 0 \ \mathbb{P}\text{- a.s} \qquad \mathbb{E}(W_t) = 0 \ \forall t \geq 0 \qquad \mathbb{E}(W_t^2) = t$$

In this thesis we will always assume a Brownian motion to be standard. Note that for a standard Brownian motion W we have $V(W_t) = \mathbb{E}(W_t^2) - \mathbb{E}(W_t)^2 = \mathbb{E}(W_t^2) = t$.

Also that this way of defining Brownian motion already gives us the **Markov Property**: For every $s \geq 0$ we have that the process $\tilde{W} := W_{t+s} - W_s$ is again a Brownian motion which is independent of $\mathcal{F}_s$, i.e independent of all the values the original Brownian motion W took before time s.

More detailed information about the Markov Property for Brownian motion can be found in Karatzas & Shreve [8], who dedicated the sections 2.5 and 2.6 to this topic.

Very important for this thesis is the concept of stopping times, which should already be familiar to the reader. We recall the definition and some basic properties from Lamberton [9, Section 3.1].

Definition 1.2.
A **stopping time** with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ is a random variable τ with values in $\mathbb{R}^+ \cup \{\infty\}$, such that for any $t \geq 0$ we have

$$\{\tau \leq t\} \in \mathcal{F}_t.$$

The associated σ-algebra is defined as

$$\mathcal{F}_\tau := \{A \in \mathcal{F} \mid \forall t \geq 0 : A \cap \{\tau \leq t\} \in \mathcal{F}_t \}.$$

Theorem 1.3.
Let S and T be two stopping times for the filtration $(\mathcal{F}_t)$. Then:

i) S is $\mathcal{F}_S$ measurable.

ii) If S is finite a.s and $(X_t)_{t\geq 0}$ is a continuous adapted process, then X_S is $\mathcal{F}_S$ measurable.

iii) If $S \leq T$ a.s, then $\mathcal{F}_S \subseteq \mathcal{F}_T$.

iv) $S \wedge T$ is a stopping time.

Note that, as any fixed $t \geq 0$ is a stopping time, property iv) implies that $S \wedge t$ is a stopping time. This will be very useful later.

A good example for a stopping time is the hitting time of a Brownian motion of a closed set $A \subseteq \mathbb{R}$:

$$\tau_A := \inf\{t \geq 0 \ : W_t \in A\}$$

One can find a proof of this claim in [10] [Le Gall, page 49].

1.2 Martingales and the Optional Sampling Theorem

In this section we mostly use Le Gall [10] and Lamberton [9] to introduce the concept of martingales.

Definition 1.4.
We call the stochastic process $M = (M_t)_{t \geq 0}$ a **martingale** with respect to the filtration $(\mathcal{F}_t)$, if...

i) ...M_t is integrable at all times, i.e $\forall t \geq 0 : \; \mathbb{E}(|M_t|) < \infty$.

ii) ...M is adapted to $\mathcal{F}$, i.e $\forall t \geq 0 : M_t$ is $\mathcal{F}_t$ measureable.

iii) ...$\forall s \leq t : \; \mathbb{E}(M_t|\mathcal{F}_s) = M_s$

If in iii) we have $\mathbb{E}(M_t|\mathcal{F}_s) \geq M_s$, we call M a **submartingale**. If we have $\mathbb{E}(M_t|\mathcal{F}_s) \leq M_s$, we call M a **supermartingale**.

Loosely speaking, a martingale is a process which is expected to be fair on average. Submartingales are expected to increase on average, and supermartingales are expected to decrease on average.

Example 1.5 ([10], Page 50).
A Brownian motion (W_t) is always a martingale since it is adapted, integrable and for every $s \geq 0$ the increments $W_t - W_s$ are independent from $\mathcal{F}_s$:

$$\mathbb{E}(W_t \mid \mathcal{F}_s) - W_s = \mathbb{E}(W_t - W_s \mid \mathcal{F}_s) = \mathbb{E}(W_t - W_s) = \mathbb{E}(W_t) - \mathbb{E}(W_s) = 0 - 0 = 0$$

The first equality follows from the fact that W_s is $\mathcal{F}_s$-measurable.

One can also show that the process $(W_t^2 - t)$ is a martingale as well:

Example 1.6.
Let W be a Brownian motion with respect to $\mathcal{F}_t$. Then the process $M_t := W_t^2 - t$ is an $\mathcal{F}_t$-martingale. To show this, observe that (M_t) is clearly adapted and integrability follows from

$$\mathbb{E}(|W_t^2 - t|) \leq \underbrace{\mathbb{E}(W_t^2)}_{t} + \underbrace{\mathbb{E}(|t|)}_{t} = 2t.$$

It remains to show property iii). Remember that $W_t = W_t - W_0$ has the same law as $W_{t+s} - W_s$ by property iii) of Brownian motion (see Definition 1.1), and that $\tilde{W}_t =$

$W_{t+s} - W_s$ is a Brownian motion, independent of $\mathcal{F}_s$, by Markov Property:

$$\mathbb{E}(W_t^2|\mathcal{F}_s) = \mathbb{E}(\underbrace{(W_{t+s} - W_s)^2}_{\tilde{W}_t}|\mathcal{F}_s) = \mathbb{E}(\tilde{W}_t^2) = t$$

Therefore, we have that

$$\mathbb{E}(W_t^2 - t|\mathcal{F}_s) = \mathbb{E}(W_t^2|\mathcal{F}_s) - t = t - t = 0,$$

and $M_t = W_t^2 - t$ is a martingale.

Example 1.7.
Let $Z \in L^1$ be an integrable random variable, and let $(\mathcal{F}_t)$ be a filtration. Then the process $M_t := \mathbb{E}(Z \mid \mathcal{F}_t)$ is a martingale. To show this, note that (M_t) is integrable because by definition of conditional expectation (see SHREVE [17, 2.3.1, Page 68]) and conditional Jensen (see SHREVE [17, 2.3.2, Page 70]) we have

$$\mathbb{E}(|M_t|) = \mathbb{E}\big(|\mathbb{E}(Z \mid \mathcal{F}_t)|\big) \leq \mathbb{E}\big(\mathbb{E}(|Z| \mid \mathcal{F}_t)\big) = \mathbb{E}(|Z|) < \infty.$$

Further, note that by *Iterated conditioning* (see also SHREVE [17, 2.3.2, Page 70]) we have for $s \leq t$:

$$\mathbb{E}(M_t \mid \mathcal{F}_s) = \mathbb{E}\big(\mathbb{E}(Z \mid \mathcal{F}_t) \mid \mathcal{F}_s\big) = \mathbb{E}(Z \mid \mathcal{F}_s) = M_s$$

One very important theorem we'll need later is the Optional Sampling Theorem (OST), which we can find in LAMBERTON [9, Section 3.3]:

Theorem 1.8 (Optional Sampling Theorem, OST).
Let $(M_t)_{t\geq 0}$ be a continuous martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, and let τ_1 and τ_2 be two stopping times such that

$$\tau_1 \leq \tau_2 \leq K$$

for some finite real number K. Then M_{τ_2} is integrable, and

$$\mathbb{E}(M_{\tau_2} \mid \mathcal{F}_{\tau_1}) = M_{\tau_1}$$

A more detailed version of this theorem and the proof can be found in KARATZAS & SHREVE [8, Section 1.3/C].

Note that this theorem implies that for any bounded stopping time τ we have

$$\mathbb{E}(M_\tau) = \mathbb{E}(M_0),$$

and in most cases (because one often assumes $\mathcal{F}_0 = \{\emptyset, \Omega\}$) we even get $\mathbb{E}(M_\tau) = \mathbb{E}(M_0) = M_0$ because M_0 needs to be $\mathcal{F}_0$ measureable, hence constant.

Remark 1.9.
There is also a version of the sampling theorem for submartingales M_t. In this case we have

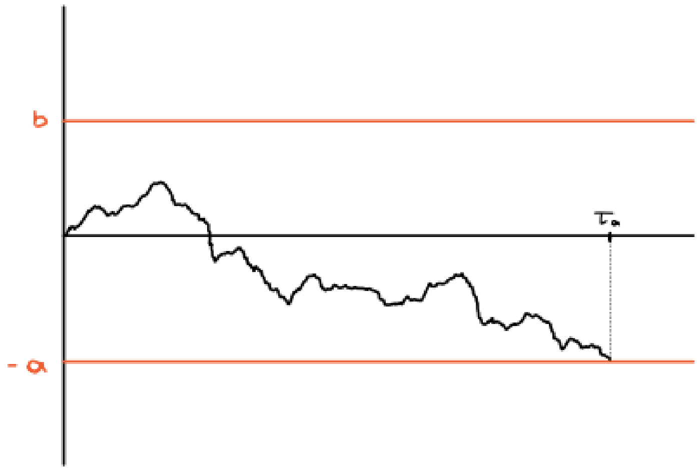
$$\mathbb{E}(M_{\tau_2} \mid \mathcal{F}_{\tau_2}) \geq M_{\tau_1} \ a.s$$

One finds this in LAMBERTON [9, Remark 3.3.5].

With help of the OST we can show some interesting results about Brownian motion. First we focus on the classical problem where we lock Brownian motion in an interval $[-a, b]$ and compute for each boundary $-a$ and b the probability for the Brownian motion to hit it before it hits the other one.

Theorem 1.10.
Let $(W_t)_{t\geq 0}$ be a Brownian motion, and let $-a < 0 < b$. Let $\tau_a := \inf\{t > 0 \mid W_t = -a\}$ and $\tau_b := \inf\{t > 0 \mid W_t = -b\}$ be two hitting times. Then:

$$\mathbb{P}(\tau_b < \tau_a) = \frac{a}{a+b}$$



Proof.
First observe that τ_a and τ_b are hitting times of closed sets and therefore stopping times. Further, the Markov Property allows us to shift the given interval $[-a, b]$ by a to $[0, a+b]$, to make our computations easier. Note that now we need to assume that our Brownian motion starts from a instead of 0.

Let $T := \tau_a \wedge \tau_b$, i.e we stop whenever Brownian motion hits the boundary, no matter which one. As T is not bounded, and we would like to apply OST (1.8), let us work with the stopping time $T \wedge n$ instead, where n is a natural number. The OST now gives us

$$\mathbb{E}(W_{\tau\wedge n}) \overset{OST}{=} \mathbb{E}(W_0) = a.$$

Our final argument is the Dominated Convergence Theorem (see Theorem 5.1):

$$\mathbb{E}(W_T) = \mathbb{E}(\lim_{n\to\infty} W_{T\wedge n}) \overset{DCT}{=} \lim_{n\to\infty} \underbrace{\mathbb{E}(W_{T\wedge n})}_{a} = a$$

Note that the use of DCT is allowed since $|W_t| \leq a + b$ for any $t \leq T$. Now we can compute:

$$a = \mathbb{E}(W_T) = \mathbb{E}(0 \cdot \mathbb{1}_{\tau_0 < \tau_{a+b}} + (a+b) \cdot \mathbb{1}_{\tau_{a+b} < \tau_0}) = (a+b) \cdot \mathbb{P}(\tau_{a+b} < \tau_0)$$

The desired result follows directly.

□

We can also investigate the expected amount of time Brownian motion needs to hit the boundaries:

Theorem 1.11.
Let $(W_t)_{t\geq 0}$ be a Brownian motion. Then:

$$\mathbb{E}(\tau_a \wedge \tau_b) = a \cdot b$$

Proof.
Let $T := \tau_a \wedge \tau_b$ be as above. We apply the OST (1.8) to the martingale $M_t := W_t^2 - t$ (see Example 1.6 and the stopping time $T \wedge n$:

$$0 = \mathbb{E}(M_0) \overset{OST}{=} \mathbb{E}(M_{T\wedge n}) = \mathbb{E}(W_{T\wedge n}) - \mathbb{E}(T \wedge n)$$

We continue by using Monotone Convergence Theorem (see Theorem 5.2) and Dominated Convergence Theorem (see Theorem 5.1):

$$\mathbb{E}(T) = \mathbb{E}(\lim_{n\to\infty} T \wedge n) \overset{MCT}{=} \lim_{n\to\infty} \mathbb{E}(T \wedge n) = \lim_{n\to\infty} \mathbb{E}(W_{T\wedge n}^2) \overset{DCT}{=} \mathbb{E}(\lim_{n\to\infty} W_{T\wedge n}^2) = \mathbb{E}(W_T^2)$$

This expectation can be computed with help of theorem 1.10:

$$\mathbb{E}(W_T^2) = \frac{a}{a+b} \cdot b^2 + \frac{b}{a+b} \cdot a^2 = a \cdot b$$

□

Another important result we will use later is *Wald's Lemma*:

Lemma 1.12 (Wald's first Lemma).
Let τ be an integrable stopping time, i.e $\mathbb{E}(\tau) < \infty$. Then $\mathbb{E}(W_\tau) = 0$.

Lemma 1.13 (Wald's second Lemma).
Let τ be an integrable stopping time, i.e $\mathbb{E}(\tau) < \infty$. Then we have $\mathbb{E}(W_\tau^2) = \mathbb{E}(\tau)$

One can find these results and the proofs in MÖRTERS [11, Page 55].

1.3 The Stochastic Integral

In this section you find a short overview of some basic terminology regarding the stochastic integral. We will introduce for instance variation, quadratic variation, local martingales, the spaces $\mathcal{M}_c^2$ and $H(M)$, and of course some different versions of the integral itself.

1.3.1 Finite Variation Integral

As a guideline we use Le Gall [10, Chapter 4].

Definition 1.14.
Let $T \subseteq \mathbb{R}$. We call $f : T \to \mathbb{R}$ a **càdlàg function**, if $\forall t \in T$ we have

$$\lim_{s \searrow t} f(s) = f(t)$$

and $\lim_{s \nearrow t} f(s)$ exists. We call a stochastic process (X_t) a **càdlàg process**, if the paths $t \mapsto X_t(\omega)$ are càdlàg functions a.s.

Càdlàg functions and processes play a very important role in stochastic analysis, especially in the build up of the stochastic integral or for martingales. In this thesis, we mostly work with continuous functions/processes, which are trivially càdlàg.

Definition 1.15.
Let $a : [0, \infty) \to \mathbb{R}$ be a nondecreasing càdlàg function, and let $da([s,t)) := a(t) - a(s)$ be a measure. We define the **Lebesgue-Stieltjes integral** of a function f with respect to a as:

$$(f \cdot a)(t) := \int_0^t f(s)\ da(s)$$

If $a = a_1 - a_2$ can be written as difference of two nondecreasing càdlàg functions (but is not necessarily càdlàg or nondecreasing itself!) we define

$$(f \cdot a)(t) := (f \cdot a_1)(t) - (f \cdot a_2)(t)$$

Definition 1.16.
Let $a : [0, \infty) \to \mathbb{R}$, $n \in \mathbb{N}$ and let

$$V_t^n := \sum_{k=0}^{\lfloor 2^n \cdot t \rfloor - 1} \left| a\left(\frac{k+1}{2^n}\right) - a\left(\frac{k}{2^n}\right) \right|.$$

We call the function $V : t \mapsto V_t := \lim_{n \to \infty} V_t^n$ the **total variation** of a.

Note that if a is an increasing function we always have

$$\left|a\left(\frac{k+1}{2^n}\right) - a\left(\frac{k}{2^n}\right)\right| = a\left(\frac{k+1}{2^n}\right) - a\left(\frac{k}{2^n}\right)$$

such that

$$V_t^n = a\left(\frac{\lfloor 2^n \cdot t \rfloor}{2^n}\right) - a(0) < \infty$$

Therefore, an increasing function always has finite total variation.

We define the total variation of a stochastic process with the same idea. The only difference is that we do not use the values of a given function, but the (random) values of the stochastic process. One can think of the Total Variation of a process as the Total Variation of the function defined by the (random) path of the stochastic process.

Definition 1.17.
Let $(A_t)_{t\geq 0}$ be a càdlàg adapted process. We define the **variation process** $(V_t)_{t\geq 0}$ by $V_t(\omega) := \lim_{n\to\infty} V_t^n(\omega)$, where

$$V_t^n(\omega) = \sum_{k=0}^{\lfloor 2^n \cdot t \rfloor - 1} \left| A_{\frac{k+1}{2^n}}(\omega) - A_{\frac{k}{2^n}}(\omega) \right|$$

If $V_t < \infty$ we say that A has finite total variation in $[0, t)$.

Definition 1.18.
Let (A_t) be a càdlàg adapted stochastic process with finite total variation in $[0, t)$, and let (H_t) be a stochastic process such that

$$\forall t \geq 0 \ \forall \omega \in \Omega : \int_0^t |H_s(\omega)| dA_s(\omega) \leq \infty.$$

Then we define the **finite variation integral** of H w.r.t A as

$$(H \cdot A)_t(\omega) := \int_0^t H_s(\omega) \ dA_s(\omega).$$

Remark 1.19.
Note that for fixed ω this is nothing but the Lebesgue-Stieltjes Integral of the function $t \mapsto H_t(\omega)$ with respect to the function $t \mapsto A_t(\omega)$. Readers might notice that this is only well defined if $t \mapsto A_t(\omega)$ is either a nondecreasing càdlàg function, or at least can be written as the difference of such. However, one can show that this is the case if and

only if (A_t) is a process of finite variation (see LE GALL [10, Page 74]). Therefore, our integral is well defined.

This type of integral has the benefit that it is not very complicated to construct, and does not need any abstract theory from stochastic analysis. Unfortunately, it requires the process against one integrates (i.e A in this chapter) to have finite total variation. This is a problem for us, as one can show that Brownian motion does not have finite variation. To prove this, one has to observe that Brownian motion is a local martingale (see Definition 1.20) and argue with help of [10, Theorem 4.8]:
A continuous local martingale (M_t) with $M_0 = 0$ a.s and finite variation (meaning the corresponding variation process V_t satisfies $V_t < \infty$ for every $t \in \mathbb{R}$) is constantly zero, i.e $M_t = 0$ for all $t \in \mathbb{R}$ a.s.

1.3.2 Local Martingales and Previsible Processes

In this chapter we follow LE GALL [10, Chapter 4] to collect some definitions we need for Itô's integral.

Definition 1.20.
We call a continuous adapted process $(M_t)_{t\geq 0}$ with $M_0 = 0$ a.s a **continuous local martingale**, if there exists a sequence $T_0 \leq T_1 \leq \dots$ of stopping times such that:

i) $T_n \nearrow \infty$, i.e $T_n(\omega) \nearrow \infty$ for every $\omega \in \Omega$.

ii) For every n the process $M^{T_n} := (M_{t\wedge T_n})_{t\geq 0}$ is a continuous martingale.

We say that $(T_n)_{n\geq 0}$ reduces M. The space of continuous local martingales will be denoted by $\mathcal{M}_{c,loc}$.

Note that any continuous martingale is a continuous local martingale, as it clearly can be reduced by the sequence $T_n = n$. The following definition will be needed when we later discuss the theorems of Girsanov (see Theorem 1.42) and Dambis-Dubin-Schwarz (see Theorem 3.3). First, we need the following lemma:

Lemma 1.21.
Let M be a continuous local martingale. Then there exists a unique increasing process $([M]_t)_{t\geq 0}$ such that the process $M_t^2 - [M]_t$ is a continuous local martingale. Further, we have that

$$[M]_t = \lim_{\max|t_{k+1}-t_k|\to 0} \sum (M_{t_k+1} - M_{t_k})^2$$

In a more precise formulation: For fixed t take an increasing sequence $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t$ of subdivisions of the interval $[0,t]$. The notation from above means

$$[M]_t = \lim_{n\to\infty} \sum_{i=1}^{p_n} (M_{t_i^n} - M^n_{t_{i-1}})^2.$$

Definition 1.22.
The process $[M]_t$ from Lemma 1.21 is called the **Quadratic Variation** of the continuous local martingale (M_t).

As for any Brownian motion (W_t) we know that $(W_t^2 - t)$ is a continuous martingale (see Example 1.6), uniqueness gives us $[W]_t = t$.

Definition 1.23.
We call a continuous martingale M **bounded in** L^2, if

$$\sup_{t\geq 0} \|M_t\|_2 = \sup_{t\geq 0} \left(\int_\Omega M_t^2 \; d\mathbb{P} \right)^{\frac{1}{2}} < \infty.$$

The space of L^2 bounded martingales will be denoted by $\mathcal{M}^2$, and the space of continuous L^2 bounded martingales will be denoted by $\mathcal{M}_c^2$

Definition 1.24.
Let M and N be two continuous local martingales. The **covariation** of M and N is the stochastic process

$$[M,N]_t := \frac{1}{2} \cdot \big([M+N, M+N]_t - [M,M]t - [N,N]_t \big).$$

In Le Gall [10, Chapter 4.4] one can find many interesting properties about covariation. The following property might help to gain some intuition:

$$[M,N]_t = \lim_{\max|t_{k+1}-t_k|\to 0} \sum (M_{t_k+1} - M_{t_k}) \cdot (N_{t_k+1} - N_{t_k}).$$

Further, one might note that the mapping $(M,N) \mapsto [M,N]_t$ is a symmetric bilinear-form. This observation is a good entry point for analysis: One can study the structure of $\mathcal{M}_c^2$ and look for interesting theorems based on theory from functional analysis. Interested readers may read Le Gall [10, Chapter 4].

The following definition can (in a little different formulation) be found in Le Gall [10, Page 43].

Definition 1.25.
We call the σ-algebra $\mathcal{P}$ generated by the sets $\{A \times (s,t] \mid s < t, A \in \mathcal{F}_s\}$ and $\{A \times \{0\} \mid A \in \mathcal{F}_0\}$ the **previsible σ-algebra**. A process (H_t) is called **previsible**, if it is $\mathcal{P}$ measurable, i.e if the mapping $H : \Omega \times [0, \infty) \to \mathbb{R}$ is $\mathcal{P}$ measurable.

Intuitively, a previsible process is a process where you know the value H_t *just a little* before time t. This will become useful later when we talk about trading strategies: People trading with stock need to decide the amount of stock they want to hold at time t a *little bit* before time t, even though this amount depends on the current stock price and therefore on randomness.

Example 1.26.
Left continuous and right continuous adapted processes are previsible, as we can read in Le Gall [10, Proposition 3.4, Page 43].

1.3.3 Itô's Integral

A rigorous construction of Itô's integral would require terminology from functional analyis, and lots of abstract argumentation. We will only give a brief introduction and explain the basic ideas. In the end we will get a type of integral which also allows us to integrate against Brownian motion, and also is compatible with the finite variation integral. This section summarizes [10, Chapter 5].

Our strategy is to define the Itô-Integral first for $M \in \mathcal{M}_c^2$ against simple processes (see Definition 1.27), and then use theory from functional analysis to define it for $M \in \mathcal{M}_c^2$ against previsible processes.

Definition 1.27.
We call a stochastic process $(H_t)_{0 \leq t \leq T}$ **simple**, if $\exists n \in \mathbb{N}$, $\exists t_1 \leq \cdots \leq t_n \leq T$ such that we can write

$$H_t(\omega) = \sum_{k=0}^{n-1} Z_k(\omega) \cdot \mathbb{1}_{(t_k, t_{k+1}]}(t),$$

where Z_k is $\mathcal{F}_{t_k}$ measurable and bounded for all $k = 1, \dots n$. We denote the set of all simple processes by $\mathcal{S}$.

Definition 1.28.
Let $H = H_t(\omega) = \sum_{k=0}^{n-1} Z_k(\omega) \cdot \mathbb{1}_{(t_k,t_{k+1}]}(t)$ be a simple process, and let $M \in \mathcal{M}_c^2$.

The **Itô-Integral** of (H_t) against (M_t) is defined as the process

$$I(H)_t := (H.M)_t = \sum_{k=0}^{n-1} Z_k \cdot \big(M_{t_{k+1}\wedge t} - M_{t_k \wedge t} \big)$$

Now we are ready to define Itô's integral against previsible processes. This requires some argumentation with functional analysis, and also the introduction of new measures and spaces. Interested readers can (as always) get a detailed and rigorous build up of this theory in Le Gall [10, Chapter 5]. In this thesis we will just give a short overview of the basic terms and results without any justification and without explaining everything properly.

For $(M_t) \in \mathcal{M}_c^2$ we define a measure μ on the previsible sigma-algebra $\mathcal{P}$:

$$\mu\big(A \times (s,t]\big) := \mathbb{E}\big(\mathbb{1}_A \cdot ([M]_t - [M]_s)\big)$$

For a previsible process $H \geq 0$ one can show that

$$\int H d\mu = \mathbb{E}\Big(\int_0^\infty H_s \; d[M]_s \;\Big),$$

where we use a finite variation integral inside of the expectation. To make sure this is well defined you need to remember that $([M]_t)$ is an increasing process (Lemma 1.21) and is therefore of finite variation (as we mentioned right after definition 1.16).

Now for our fixed (M_t) we define the space

$$L^2(M) := L^2(\Omega \times [0,\infty), \mathcal{P}, \mu),$$

where we can define the norm

$$\|H\|_M := \Big(\mathbb{E}\Big(\int_0^\infty H_s^2 \; d[M]_s\Big)\Big)^{\frac{1}{2}}.$$

One can show that $L^2(M)$ is nothing but the space of all previsible processes (H_t) with $\|H\|_M < \infty$.

Last but not least, we use the fact that the space of simple functions $\mathcal{S}$ is dense in $L^2(M)$, and extend the map I (which is currently defined on $\mathcal{S}$) to $L^2(M)$: One can show that there exists a unique embedding $I : L^2(M) \to M_c^2$ such that for $H \in \mathcal{S}$ we always have $I(H) = H \cdot M$, i.e the Itô Integral for previsible processes is consistent with the definition of the integral for simple processes. From this fact we can also take that the Itô-Integral is an L^2 bounded continuous martingale.

1.4 Itô's Formula

A very important tool in mathematical finance and in stochastic analysis is Itô's formula. In this section we use KARATZAS & SHREVE [8, Section 3.3] and REVUZ & YOR [13, Chapter IV, § 3]. First, we need the following definition:

Definition 1.29.
We call a stochastic process $(X_t)_{t\geq 0}$ a **continuous semimartingale**, if it has the decomposition

$$X_t = X_0 + M_t + B_t \quad a.s,$$

where M_t is a continuous local martingale, and $B_t = A_t^+ - A_t^-$ is the difference of two nondecreasing adapted processes. We already mentioned that (B_t) can be written as such if and only if it has finite total variation. To summarize this, a continuous semimartingale consists of two parts: a continuous local martingale, and a process of finite variation.

The covariation of two continuous semimartingales $X = X_0+M+B$ and $Y = Y_0+M'+B'$ is given by the covariation of the local martingale parts:

$$[X,Y]_t := [M,M']_t$$

This can be justified by showing that

$$\lim_{\max|t_{k+1}-t_k|\to 0} \sum (X_{t_k+1} - X_{t_k}) \cdot (Y_{t_k+1} - Y_{t_k}) = [M,M']_t$$

The intuition behind this is the observation that B does not contribute to quadratic variation, as it is of finite total variation. A proof of this can be found in [10] [LEGALL, SECTION 4.5].

1.4.1 Itô's Formula: Simple Version

First, we introduce Itô's formula for the *simple* case where we restrict ourselves to only one dimension of time and only one dimension for space. Further, we only consider local martingales here for simplicity.

Theorem 1.30 (Itô's formula, simple version).
Let $M \in \mathcal{M}_{c,loc}$, and let $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be a $C^{1,2}$ function, i.e once differentiable w.r.t the first component t and twice differentiable w.r.t the second component x. Then:

$$f(t,M_t) \;=\; f(0,M_0) + \int_0^t f_t(s,M_s)\; ds + \int_0^t f_x(s,M_s)\; dM_s + \frac{1}{2}\cdot\int_0^t f_{xx}(s,M_s)\; d[M_s],$$

where $f_t(t,x) = \frac{\partial}{\partial t} f(t,x)$ and $f_x(t,x) = \frac{\partial}{\partial x} f(t,x)$ denote the partial derivatives of f.

Readers who are familiar with stochastic calculus are used to the notation

$$df(t, M_t) = f_t(t, X_t)\ dt + f_x(t, M_t)\ dM_t + f_{xx}(t, M_t)\ d[M]_t.$$

Remark 1.31.
Note that we have three different types of integrals here:

- The integral $\int_0^t f_t(s, X_s)\ ds$ is the Riemann integral of a continuous function over an interval.
- The integral $\int_0^t f_x(s, X_s)\ dM_s$ is the Itô-Integral defined in the last section. It is important to observe that the process $(f_x(t, M_t))_{t\geq 0}$ is continuous (as (M_t) is continuous and $f \in C^{1,2}$), and continuous processes are previsible (see Example 1.26).
- The integral $\int_0^t f_{xx}(s, X_s)\ d[M]_s$ is the Lebesgue-Stieltjes integral of $t \mapsto \frac{\partial^2}{\partial x^2} f(t, X_t)$ against the process $[M]_t$, which is nondecreasing and therefore of finite total variation (see Remark 1.19).

All of these integrals depend on the path $t \mapsto X_t(\omega)$ and therefore on randomness.

If one takes a Brownian motion (W_t) for the local martingale, it is useful to remember the fact $[W]_t = t$ and conclude $d[W]_t = dt$ (see right under definition 1.22).

Example 1.32.
Let W_t be a Brownian motion, and let

$$X_t := e^{(\mu - \frac{\sigma^2}{2})\cdot t + \sigma \cdot W_t},$$

where μ and σ are real valued constants. We apply Itô's formula to $f(t, x) = e^{(\mu - \frac{\sigma^2}{2})\cdot t + \sigma\cdot x}$ (such that $X_t = f(t, W_t)$) and get:

$$\begin{aligned} dX_t &= \underbrace{(\mu - \frac{\sigma^2}{2}) \cdot e^{(\mu - \frac{\sigma^2}{2})\cdot t + \sigma \cdot W_t}}_{f_t(t, W_t)}\ dt + \underbrace{\sigma \cdot e^{(\mu - \frac{\sigma^2}{2})\cdot t + \sigma \cdot W_t}}_{f_x(t, W_t)}\ dW_t + \frac{1}{2} \cdot \underbrace{\sigma^2 e^{(\mu - \frac{\sigma^2}{2})\cdot t + \sigma \cdot W_t}}_{f_{xx}(t, W_t)}\ dt \\ &= e^{(\mu - \frac{\sigma^2}{2})\cdot t + \sigma \cdot W_t} \cdot \big((\mu - \frac{\sigma^2}{2})dt + \sigma\ dW_t + \frac{\sigma^2}{2}\ dt\big) \\ &= X_t \cdot (\mu\ dt + \sigma\ dW_t) \end{aligned}$$

Itô's formula can also be used as a tool to show that a process is a local martingale:

Example 1.33.
Let $Z_t := e^{-\theta\cdot W_t-\frac{1}{2}\theta^2 t} = f(t,W_t)$ for $f(t,x) = e^{-\theta x-\frac{1}{2}\theta^2\cdot x}$. We apply Itô to get

$$\begin{aligned} dZ_t &= -\frac{1}{2}\cdot\theta^2\cdot Z_t\ dt - \theta\cdot Z_t\ dW_t + \frac{1}{2}\cdot\theta^2\cdot Z_t\ dt \\ &= -\theta\cdot Z_t\ dW_t, \end{aligned}$$

and conclude

$$Z_t = \underbrace{1}_{Z_0} - \theta\cdot\underbrace{\int_0^t Z_u\ dW_u}_{\text{martingale}},$$

which is a local martingale since the stochastic integral is a local martingale (see the end of page 23).

1.4.2 Itô's Formula: General Version

Theorem 1.34 (Itô's formula: general version).
Let $X_t = (X_s^1, ..., X_s^d)$ be a vector of continuous semimartingales and let $f \in C^2(\mathbb{R}^d, \mathbb{R})$. Then:

$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial}{\partial x_i} f(X_s)\ dX_s^i\ +\ \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial^2}{\partial x_i x_j} f(X_s)\ d[X_s^i, X_s^j]$$

This theorem is a generalization of the *simple* version in many ways: First of all, we do not restrict ourselves to local martingales and work with semimartingales instead. Further, it allows us to use functions depending on more than one stochastic process, i.e it is clearly a multi dimensional formula. Also, observe that there is no extra term for time, so the function f does not necessarily take t as an argument. Of course one can choose $X_t^j = t$ anyway to add a component for time.
Itô's formula can also be written in *differential notation*:

$$df(X_t) = \sum_{i=1}^n \frac{\partial}{\partial x_i} f(X_t)\ dX_t^i\ + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial}{\partial x_i x_j} f(X_t)\ d[X^i, X^j]_t$$

Example 1.35 ([13], Page 146).
Let X and Y be continuous local martingales. We want to compute $d(X_tY_t)$ using Itô's

formula. Let $f(x,y) = xy$:

$$\begin{aligned} df(X_t, Y_t) &= f_x(X_t, Y_t)\ dX_t + f_y(X_t, Y_t)\ dY_t + \frac{1}{2} \cdot 2 \cdot f_{xy}(X_t, Y_t)\ d[X,Y]_t \\ &= Y_t\ dX_t + X_t\ dY_t + d[X,Y]_t \end{aligned}$$

If one sets $X = Y$ this formula implies

$$dX_t^2 = 2 \cdot X_t\ dX_t + d[X]_t,$$

which reads as follows in *integral form*:

$$X_t^2 = X_0^2 + 2 \cdot \int_0^t X_s\ dX_s + [X]_t$$

This does not only give us an expression for the local martingale $X_t^2 - [X]_t$, but also a possibility to compute $[X]_t$:

$$[X]_t = X_t^2 - X_0^2 - 2 \cdot \int_0^t X_s\ dX_s$$

1.4.3 Itô Processes

In this section we introduce Itô processes. It is easy to compute the quadratic variation of this special type of process, which will be useful later. We use LAMBERTON [9, Section 3.4.2].

Definition 1.36.
Let (W_t) be a Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, and let (X_t) be a stochastic process on the same probability space. We call (X_t) an **Itô process**, if it can be written as

$$X_t = X_0 + \int_0^t K_s\ ds\ + \int_0^t H_s\ dW_s \qquad \mathbb{P} \text{ a.s},$$

where..

i) ..X_0 is $\mathcal{F}_0$ measurable,

ii) (K_t) and (H_t) are $\mathcal{F}_t$ adapted,

iii) $\int_0^t |K_s| ds < \infty$ $\mathbb{P}$ a.s and

iv) $\int_0^t |H_s|^2 ds < \infty$ $\mathbb{P}$ a.s.

One can show that this decomposition is unique (see LAMBERTON [9, Page 66]), and that the quadratic variation of an Itô process X_t is given by

$$[X]_t = \int_0^t H_s^2 \, ds,$$

which can be found in LAMBERTON [9, Page 67].

1.5 Change of Measure: The Radon–Nikodym Theorem

In this section we explain the concept of changing a measure and introduce the theorem of Radon-Nikodym. Further, we shortly explain how this will be useful later when we talk about pricing options. We summarize SHREVE [17, Section 1.6].

The first important information in this section is that, if Z is a nonnegative random variable with $\mathbb{E}(Z) = 1$, the set function defined by

$$\mathbb{Q}(A) := \int_A Z \, d\mathbb{P}$$

is a probability measure. All we do here is to reassign probabilities in Ω according to Z: If $Z > 1$, the probabilities grow, and if $Z < 1$ the probabilities decrease. Very important is here to note that nullsets of $\mathbb{P}$ are also nullsets of $\mathbb{Q}$ and vice versa. Whenever measures $\mathbb{P}$ and $\mathbb{Q}$ satisfy this condition, we call them **equivalent**.

The reassignment of probabilities gives us the possibility to change the distribution of a random variable without changing the random variable itself. Computations in this setting can be easier and very useful, as we will see in the next section.

The second important information here is that there exists a relationship between the expectations $\mathbb{E}_{\mathbb{Q}}$ (w.r.t $\mathbb{Q}$) and $\mathbb{E}$ (w.r.t $\mathbb{P}$). For any random variable X defined on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ we have:

$$\mathbb{E}_{\mathbb{Q}}(X) = \mathbb{E}(XZ)$$

This relationship is trivial for indicator functions $\mathbb{1}_A$:

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_A) = \mathbb{Q}(A) = \int_\Omega \mathbb{1}_A \cdot Z \, d\mathbb{P} = \mathbb{E}(\mathbb{1}_A \cdot Z),$$

and can be extended to simple functions $g = \sum_{k=1}^n \alpha_k \cdot \mathbb{1}_{A_k}$ by linearity. The final argument is the fact that measurable functions can be approximated by an increasing sequence of simple functions, and from Monotone Convergence Theorem (Theorem 5.2). For details see SHREVE [17]. Until now we were focusing on the situation that you have one probability measure $\mathbb{P}$ and one *reassigning function* Z to get an equivalent measure $\mathbb{Q}$. The theorem of Radon–Nikodym tells us that for equivalent probability measures $\mathbb{P}$ and $\mathbb{Q}$ we can always find such a random variable:

Theorem 1.37 (Radon–Nikodym, [17], Page 39).
Let $\mathbb{P}$ and $\mathbb{Q}$ be equivalent probability measures defined on $(\Omega, \mathcal{F})$. Then there exists a nonnegative random variable Z such that $\mathbb{E}_{\mathbb{P}}(Z) = 1$ and

$$\mathbb{Q}(A) = \int_A Z \, d\mathbb{P}.$$

The random variable Z will be denoted by $\frac{d\mathbb{Q}}{d\mathbb{P}}$ and is called the **Radon-Nikodym derivative**.

Note that the property $\mathbb{E}_{\mathbb{Q}}(X) = \mathbb{E}(X \cdot Z)$ with this notation (and with integrals instead of expectations) reads

$$\int_\Omega X \; d\mathbb{Q} = \int_\Omega X \cdot \frac{d\mathbb{Q}}{d\mathbb{P}} \; d\mathbb{P},$$

and obviously explains the notation $\frac{d\mathbb{Q}}{d\mathbb{P}}$ for Z.

Note that if one has $\mathbb{P} = \lambda$, where λ denotes the Lebesgue measure on $\mathbb{R}$, the random variable Z is a real valued function $\mathbb{R} \to \mathbb{R}$.

Let X be a (real valued) random variable with density f and law $\mathbb{Q}$. In this case, standard probability theory tells us

$$\mathbb{Q}((-\infty, x)) = \int_{-\infty}^{x} f(x) \; dx,$$

which is an integral with respect to the Lebesgue measure λ. We can observe that the Radon-Nikodym derivative in this simple case is the density we know from standard probabilty theory.

The property $\mathbb{E}_{\mathbb{Q}}(X) = \mathbb{E}(X \cdot Z)$ from above now reads as follows:

$$\int_{-\infty}^{\infty} x \; d\mathbb{Q} = \int_{-\infty}^{\infty} x \cdot f(x) \; dx$$

Example 1.38.
Let X have exponential distribution, i.e the distribution function is $F(x) = 1 - e^{-\lambda x}$ for $x \geq 0$ with $\lambda \geq 0$. Let μ denote the distribution of X. Then X has density $\lambda \cdot e^{-\lambda x}$, and

$$\begin{aligned} \mathbb{E}(X) &= \int_0^\infty x \; d\mu(x) = \int_0^\infty x \cdot \underbrace{f(x)}_{\lambda \cdot e^{-\lambda x}} \; dx = \lambda \cdot \Bigg(\underbrace{-\frac{1}{\lambda} e^{-\lambda x} \cdot x \Big|_0^\infty}_{0} + \int_0^\infty \frac{1}{\lambda} e^{-\lambda x} \; dx \Bigg) \\ &= \int_0^\infty e^{-\lambda x} dx = -\frac{1}{\lambda} \cdot \underbrace{\left(e^{-\lambda x} \Big|_0^\infty \right)}_{-1} = \frac{1}{\lambda} \end{aligned}$$

1.6 Terminology from Mathematical Finance

In this section we introduce terminology as in Lamberton [9] and describe the classical approach to mathematical finance. We also use Shreve [17] and the first chapter of Hobson [7].
We are modelling the price of some given stock (or asset) with a stochastic process $(P_t)_{0 \leq t \leq T}$ defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$. The time period $[0, T]$ on which we observe the stock is assumed to be short enough such that it is reasonable to assume that the **interest rate** is a constant $r > 0$.
The process $(S_t)_{0 \leq t \leq T}$ with $S_t := e^{-r \cdot t}$ is called the **discounted stock price**. It allows us to compare the value of the stock at any time with other amounts of money or with other stock prices at time 0.

The motivation for discounting P is the following: Assume that at time 0 you buy the asset P at total cost P_0. After that, the stock prices increases such that $P_T > P_0$. One might think this is a success, because selling the option at time T would yield a profit of $P_T - P_0$. However, you have to ask yourself the following question:
"How much money would I now have if I didn't invest in P and left the money on my bank account instead? ".
In this case the amount P_0 (units of money) on the bank account would grow with interest rate r and at time T we would have $e^{rT} \cdot P_0$. The *real* profit is $e^{rT} \cdot P_0 - P_0$.

To summarize this: If we want to know if you really made profit by buying the asset with price process (P_t), we need to check if $P_T > e^{rT} \cdot P_0$, i.e if at time T the discounted stock price S_T is bigger than P_0.

Altogether we call $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P}, (S_t)_{0 \leq t \leq T})$ a **market model**.

We use a process $(H_t)_{t \geq 0}$ to describe the amount of stock we hold at a time t. As we have to decide the quantity H_t infinitesimally *before* time t, we require H to be a previsible process (see Definition 1.25), and call H a **trading strategy**.

1.6.1 Options

In this thesis we will talk a lot about prices of so called options (or sometimes *contingent claims*). We will introduce this term formally later in the Black–Scholes Model (see Definition 1.47). For now it is enough to explain some examples:

The *most basic* option is the so called **European call option**

$$C^{call} = (P_T - K)^+,$$

where $X^+ = \max\{X, 0\}$. The owner of this option has the right to buy the stock at **maturity** time T for the **strike price** K. If the stock price at time T exceeds K, the owner of this option has a payoff of $P_T - K$. If this is not the case, the owner does not use this right as it would be *negative payoff*. Observe that the owner of the option only makes profit if $P_T - K$ is higher than what they paid for the option. Buying a Call option is reasonable if one thinks the stock price is going to increase, i.e if one expects $P_T > P_0$.

In this chapter we will discuss how to find *fair* prices for options by using the underlying stochastic model. In the *main part* of this thesis, we will look for a possibility to compute prices of *more complicated* options only by using prices of frequently traded options like this.

The counterpart of the Call Option is the **European put option**

$$C^{put} = (K - P_T)^+,$$

which allows the owner to sell the stock for the strike price K at maturity T. Buying a Put Option is reasonable if one expects the stock price to fall until maturity.

These two options are introduced in Shreve [17] on the pages 155 and 163. In [17] one can (among many others) also find **American options** (page 339), **Asian options** (pages 278 and 320) or the **Barrier option** (page 299).

In this thesis we will focus on European options (with maturity T), which will be defined by a nonnegative $\tilde{\mathcal{F}}_T$-measurable random variable h. The filtration $(\tilde{\mathcal{F}}_t)$ will be specified later (see Definition 1.47). One can think of $h(\omega)$ as the payoff of the option h in the scenario ω. Each $\omega \in \Omega$ determines the path $(P_t(\omega))_{t\geq 0}$, and based on this path the payoff $h(\omega)$ can be computed.

Note that the European call and put option both use only the final value P_T of the asset and do not need the values $\{P_t \mid 0 \leq t < T\}$. Options like this are called **Vanilla options**. Options where the payoff depends on the path $\{P_t \mid 0 \leq t < T\}$ of the asset are called **Exotic options**. (see Shreve [17, Page 229]). We will work with the exotic Barrier option in section 2.1.

We call an option h **replicable** (or attainable) if there exists a trading strategy $(H_t)_{t\geq 0}$ such that

$$h = (H \cdot P)_T = \int_0^T H_t \, dP_t$$

If H attains h, i.e if H satisfies the equation above, we call H a **hedging strategy**, **replicating strategy** or **attaining strategy** for the option h. Loosely speaking, buying this replicating portfolio (i.e using the hedging strategy) or buying the option h should not make a difference. Markets where every option is attainable are called **complete** . Note that, as H is previsible, the process $(H \cdot P)_t = H_t \cdot P_t$ is a continuous martingale. The following term will be important in the last chapter:

Definition 1.39.
We call H a **subreplicating strategy** for h if we have

$$h \leq (H \cdot P)_T = \int_0^T H_t \, dP_t.$$

Remark 1.40.
Not every trading strategy H can be a hedging strategy. Later we will define the class of admissible strategies, which determines some properties we need for H (see Definition 1.48).

1.6.2 Pricing Options: An Intuitive Approach

In this section we follow an intuitive approach for finding a *fair* price of a call option. We will introduce a very *simple* model and exploit that pricing options is a very complicated matter, which does (at least at first) not agree with ones intuition. We will follow Delbaen & Schachermayer [4, Chapter 1].

Let us postulate the *simple* model we just mentioned. Suppose the price of the risky asset P (which we will model in discrete time) depends on a fair coin (i.e. $\mathbb{P}(\text{Heads}) = \mathbb{P}(\text{Tails}) = \frac{1}{2}$), and suppose $P_0 = 1$:

$$P_1(\omega) = \begin{cases} 2 & \dots \quad \omega = \text{Heads} \\ \frac{1}{2} & \dots \quad \omega = \text{Tails} \end{cases}$$

From now on we will write H and T instead of Heads and Tails.

There is also the possibility to invest in a riskless asset B, which is mostly called bond or numéraire. One can think of this as putting money on a bank account. For simplicity, we assume the interest rate to be 0 and the price of one unit to be 1, i.e $B_0 = B_1 = 1$.

Let $C = (P_1 - 1)^+$ be the call option with strike price $K = 1$ and maturity $T = 1$ as we introduced in section 1.6.1. The first idea for pricing this call would probably be to compute the expectation of C, i.e to ask the following question:

> What do we expect C to be worth at time 1?

So let us anwer:

$$\mathbb{E}_{\mathbb{P}}(C) = 1 \cdot \mathbb{P}(H) + 0 \cdot \mathbb{P}(T) = \frac{1}{2}$$

Now suppose we do the following:

Step 1: At time 0 we buy $\frac{2}{3}$ units of the risky asset at cost $\frac{2}{3} \cdot P_0$.

Step 2: At time 0 we *borrow* $\frac{1}{3}$ units of the bond, which has a negative cost of $-\frac{1}{3}$.

In total we hold a portfolio (where we denote the value by Π) consisting of $\frac{2}{3}$ of the risky asset and the obligation to *return* $\frac{1}{3}$ units of the bond at time $t = 1$. Acquiring this portfolio costs $\frac{2}{3}P_0 - \frac{1}{3}B_0 = \frac{1}{3}$ because we chose $P_0 = B_0 = 1$ at the beginning.

This portfolio replicates the call option C because the value Π satisfies:

$$\Pi(H) = \underbrace{\frac{2}{3} \cdot \overbrace{2}^{=P_1(H)}}_{\text{value of what we own}} - \underbrace{\frac{1}{3} \cdot \overbrace{1}^{=B_1(H)}}_{\text{value of what we need to return}} = 1 = C(H)$$

$$\Pi(T) = \underbrace{\frac{2}{3} \cdot \overbrace{\frac{1}{2}}^{=P_1(T)}}_{\text{value of what we own}} - \underbrace{\frac{1}{3} \cdot \overbrace{1}^{=B_1(H)}}_{\text{value of what we need to return}} = 0 = C(T)$$

Step 3: We sell the call option C at time 0 at price $\frac{1}{2}$. It is not necessary to own the risky asset at the time we sell the call option. If the buyer wants to buy one unit of the stock at price $K = 1$ from us at time 1 (which is his right), we can still buy the stock at time 1.

By now we spent $\frac{1}{3}$ acquiring the portfolio with value Π and got $\frac{1}{2}$ by selling C, which yields a profit of $\frac{1}{6}$.

Step 4: In the case of *Heads* we have to buy 1 unit of the stock (at price $P_1 = 2$) and sell it to our buyer for $K = 1$, which yields a loss of $C(H) = 1$. Fortunately, our portfolio yields a profit of $\Pi(H) = 1 = C(H)$, which compensates our loss from the Call.
In the case of *Tails* the buyer of the call will not use his right to buy one unit of stock from us at price 1, as the current stock price would be $P_1 = \frac{1}{2}$. Our portfolio yields no cost and no profit in this case.

To summarize this: In any case, this strategy yields a total profit of $\frac{1}{6}$. Therefore, selling the Call at price $\frac{1}{2}$ raises the possibility to gain profit without taking any risk. By first selling the call one could also avoid making an initial investment.

Opportunities like this are called *arbitrage opportunities*. We will specify this important term later (see Definition 1.53).

In a reasonable market, there cannot be any arbitrage opportunity. That is why the call option in this section needs to be sold at price $\frac{1}{3}$. In the next section (1.6.3) the reader will notice that the original idea of using expectation to compute the price was not bad at all. Our mistake was that we used the wrong probability measure.

Remark 1.41.
Holding a negative amount of a stock or the bond (which in *real world* means borrowing and returning later) is called *going short*. Investing a positive amount money is – on the contrary – called *going long*.

1.6.3 Risk–Neutral Pricing

A very important question in mathematical finance is how much a given option costs. In this section, we explain the idea of **risk neutral pricing** and introduce Girsanov's Theorem. The guideline will be SHREVE [17, Section 1.6, Section 5.2].

To get some intuition, one could think of the space Ω as a collection of possible scenarios in our market. There is no way of knowing the exact probabilities of every scenario, therefore our probability measure $\mathbb{P}$ (which maybe comes from empirical data) is neither very reliable, nor is it convenient when it comes to computations. We call computations done with measure $\mathbb{P}$ **real world computations**.

For purposes like pricing options, we use a different concept: As we explained earlier in section 1.5, we *reassign* our probabilities with help of a (very cleverly chosen) function Z, to get a so called Risk Neutral Measure $\mathbb{Q}$. The rigorous and detailed construction of this measure will be provided by Girsanov's theorem. We call computations done with measure $\mathbb{Q}$ **risk neutral computations**.

An important requirement for our choice of $\mathbb{Q}$ is to be equivalent to $\mathbb{P}$: Scenarios that are impossible according to the real world measure $\mathbb{P}$ have to be impossible in the risk neutral world as well and vice versa.

Of course, this concept might raise the question whether the risk neutral computations really apply in the real world. For instance, the risk neutral price of an option might not be appropriate in the real world. We answer this question like SHREVE[17, Page 35] on page 35: *There is only one world.* Reassigning the probabilities does not change our view of the market, and hedges which work with measure $\mathbb{P}$ a.s also work with measure $\mathbb{Q}$ a.s. Further, remember that in the real world there is no way of knowing the exact probabilities anyway: So who is to decide which prices are appropriate?

We already mentioned that we need to choose the reassigning function in a clever way. To do this, we need to respect the so called **market price of risk** $(\theta_t)_{t\geq 0}$, which will be specified later.

Theorem 1.42 (Girsanov, [17], Page 212).
Let (W_t) be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}_t$ be a filtration for this Brownian motion. Let (θ_t) be an adapted process. Define

$$Z_t := e^{-\int_0^t \theta_u \, dW_u \, - \frac{1}{2}\int_0^t \theta_u^2 \, du}$$

and let

$$\mathbb{Q}(A) := \int_A Z_T \, d\mathbb{P}.$$

Under the assumption

$$\mathbb{E}\Big(\int_0^T \theta_u^2 Z_u^2 \, du\Big) < \infty,$$

the stochastic process $(\tilde{W}_t)$ defined by

$$\tilde{W}_t := W_t + \int_0^t \theta_u \; du$$

is a Brownian motion with respect to $\mathbb{Q}$.

Important for the proof of this theorem is the fact that the process (Z_t) is a martingale with respect to $\mathbb{P}$, which can be shown with help of Itô's formula and a little knowledge of stochastic integrals. Further, note that $\mathbb{E}(Z_T) = 1$, and therefore $\mathbb{Q}$ is a probability measure by Radon-Nikodym (see Theorem 1.37).

Of course, the question how to choose (θ_t) and why this choice justifies the name *risk neutral* is still open. A general answer is beyond the scope of this thesis, and interested readers will find their answer in SHREVE [17]. We will only specify the market price of risk for the Black–Scholes model. The name *risk-neutral* will also be explained when we use the concept of risk-neutral pricing in the Black-Scholes Model in section 1.6.4.

1.6.4 The Black–Scholes Model

Let (W_t) be a Brownian motion, and let μ, σ and $r > 0$ be real valued constants. The Black–Scholes model simulates the prices of one risky asset with price P_t and one non-risky asset P_t^0, which we will call **numeraire**. We assume the stock prices satisfy the following equations:

$$\begin{aligned} dP_t &= \mu \cdot P_t \; dt + \sigma \cdot P_t \; dW_t \\ dP_t^0 &= r \cdot P_t \; dt, \end{aligned}$$

Note that in the case $\mu = 0$ the process P_t is a martingale since the underlying equation yields

$$P_t = P_0 + \sigma \cdot \int_0^t P_u \; dW_u,$$

and the stochastic integral $(P \cdot W)_t$ is a martingale. Therefore, $\mu = 0$ means we expect stagnation of the price, $\mu > 0$ means we expect increasing prices and $\mu < 0$ means we expect decreasing prices. We call μ *drift*. One can think of σ as an index of how much P_t depends on randomness, since it controls how much changes of Brownian motion affect the stock price. We call σ *volatility*. The constant $r > 0$ controls the growth of the numeraire asset P_t^0. We can think of the numeraire as a bank account with fixed interest rate r.

From now on we assume $P_0^0 = 1$. With help of Itô's formula we showed in Example 1.32

that the equations above are solved by:

$$P_t^0 = \underbrace{P_0^0}_{=1} \cdot e^{r \cdot t}$$

$$P_t = P_0 \cdot e^{(\mu - \frac{\sigma^2}{2}) \cdot t + \sigma \cdot W_t}$$

One can think of the numeraire asset as money on a bank account, which grows by a constant interest rate r.

The **value** of our portfolio, i.e of our collection of stock, is given by

$$V_t := H_t^0 P_t^0 + H_t P_t.$$

The intuition behind the value is the following: If you want to find out how much your portfolio is worth, you have to multiply the amount of each asset you own with its current price and add up.

To understand the following part better, let us make a little excursion to discrete time models. Here it seems reasonable that the value V satisfies

$$V_{n+1} - V_n = H_{n+1}^0 \cdot \underbrace{\big(P_{n+1}^0 - P_n^0\big)}_{P_n^0 \cdot (e^r - 1)} \; + \; \underbrace{H_{n+1} \cdot \big(P_{n+1} - P_n\big)}_{\text{risky part}},$$

The value of the risky part of the portfolio increases from day n to day $n+1$ by the amount the price P increases multiplied by the amount H_{n+1} of the asset the owner holds in this period. This equation holds provided that the owner does not use any money from outside the portfolio to buy more of the risky asset or sells a part of his portfolio. If this is the case, the value V can not be described by the equation above.

Note that this approach gives us the formula

$$V_n = V_0 + \sum_{k=0}^{n-1} H_{k+1}^0 \cdot \big(P_{k+1}^0 - P_k^0\big) \; + \; \sum_{k=0}^{n-1} H_{k+1} \cdot \big(P_{k+1} - P_k\big),$$

and that we still have that

$$V_n = H_n^0 \cdot P_n^0 + H_n \cdot P_n.$$

To extend this to the continuous setting, we need the following definition:

Definition 1.43 ([9] definition 4.1.1).
A **self-financing strategy** is a pair (H^0, H) of previsible processes (H_t^0), $(H)_t$ satisfying

$$\int_0^T |H_t^0| dt \; + \; \int_0^T H_t^2 dt \; < \infty \;\; \text{a.s}$$

and

$$\underbrace{H_t^0 P_t^0 + H_t P_t}_{V_t} = \underbrace{H_0^0 P_0^0 + H_0 P_0}_{V_0} + \int_0^T H_u^0 \, dP_u \; + \; \int_0^t H_u \, dP_u$$

for all $t \in [0, T]$.

Loosely speaking, we call a strategy self-financing, if every change of the value of our portfolio comes from the stock market and not from an external source. Note that the second condition of the definition above can be written as

$$dV_t = H_t^0 dP_t^0 \; + \; H_t dP_t.$$

We also introduce the **discounted value**:

$$\tilde{V}_t = e^{-rt} \cdot V_t = H_t^0 + H_t \cdot \underbrace{e^{-rt} \cdot P_t}_{S_t}$$

Remember that the process $S_t = e^{-rt} \cdot P_t$ is called the discounted stock price.

Lemma 1.44 ([9], Lemma 4.1.2).
Let (H^0, H) be a pair of trading strategies such that

$$\int_0^T |H_t^0| dt \; + \; \int_0^T H_t^2 dt \; < \infty \text{ a.s}$$

is satisfied. Then, (H^0, H) is self-financing if and only if

$$d\tilde{V}_t = H_t dS_t$$

Proof.
Assume that (H^0, H) is self-financing. First note that by an application of Itô's formula (1.35) we get for $S_t = e^{-rt} P_t$:

$$dS_t = -r \cdot \underbrace{e^{-rt} \cdot P_t}_{S_t} \, dt \; + \; e^{-rt} \, dPt$$

The same formula also yields:

$$d\tilde{V}_t = d(e^{-rt} \cdot V_t) = -r \cdot e^{-rt} \, dt \cdot V_t + e^{-rt} dV_t$$

Note that, as (e^{-rt}) does not depend on randomness, the covariation of e^{-rt} and V_t and the covariation of e^{rt} and P_t both are zero. As our strategy is self-financing, we know

$$dV_t = H_t^0 dP_t^0 \ + \ H_t dP_t.$$

Therefore:

$$d\tilde{V}_t = -re^{-rt}\big(H_t^0 \cdot e^{rt} + H_t P_t\big)dt \ + \ e^{-rt} \cdot \big(H_t^0 dP_t^0 \ + \ H_t dP_t\big)$$

Using $dP_t^0 = -r \cdot e^{rt} dt$ and $S_t = e^{-rt} P_t$ we get

$$d\tilde{V}_t = -r \cdot H_t S_t \ dt + H_t e^{-rt} dP_t = H_t \cdot \big(\underbrace{-r \cdot S_t \ dt \ + \ e^{-rt} dP_t}_{dS_t}\big)$$

Now suppose $d\tilde{V}_t = H_t \ dS_t$ holds. The same computations show

$$dV_t = H_t^0 dP_t^0 \ + \ H_t dP_t,$$

hence (H^0, H) has to be self-financing.

□

In section 1.6.3 we explained the concept of risk-neutral pricing and introduced Girsanov's Theorem (see Theorem 1.42). Now we show how to find a risk–neutral measure $\mathbb{Q}$ in the Black–Scholes Model and how to use it to compute prices of options. One finds this in LAMBERTON [9, Page 92].

Lemma 1.45.
Define $\theta := \frac{\mu - r}{\sigma}$, let $Z_t := e^{-\theta \ W_t - \frac{1}{2}\theta^2}$ be a stochastic process and let W_t be a Brownian motion, both defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(\tilde{\mathcal{F}}_t)$ be the natural filtration of W (see Definition 5.6).

$$\mathbb{Q}(A) := \int_A Z_T \ d\mathbb{P}.$$

Then the process $(\tilde{W}_t)$ defined by $\tilde{W}_t = W_t + \theta \cdot t$ is a Brownian motion on $(\Omega, \mathcal{F}, (\tilde{\mathcal{F}}_t), \mathbb{Q})$ and we have

$$dS_t = \sigma \cdot S_t \ d\tilde{W}_t.$$

Proof.
First, we use 1.35 and remember $dP_t = P_t \cdot (\mu\ dt + \sigma\ dW_t)$ to compute

$$\begin{aligned} dS_t &= -r \cdot e^{-rt}\ P_t\ dt\ +\ e^{-rt}\ dP_t \\ &= -r \cdot \underbrace{e^{-rt}\ P_t}_{S_t}\ dt\ +\ \underbrace{e^{-rt}\ P_t}_{S_t} \cdot (\mu\ dt + \sigma\ dW_t) \\ &= S_t \cdot \big((\mu - \sigma)\ dt + \sigma\ dW_t\big) \end{aligned}$$

The assumption of Girsanov's Theorem (1.42) is satisfied as Z is a martingale by Example 1.33 and $\theta_t = \theta$ is constant. So we may apply (1.42) which yields that

$$\tilde{W}_t := W_t + \int_0^t \theta\ dt = W_t + \frac{\mu - r}{\sigma} \cdot t$$

is a Brownian motion w.r.t $\mathbb{Q}$. Now note that

$$\sigma \cdot d\tilde{W}_t = \sigma\big(dW_t + \frac{\mu - r}{\sigma}\ dt\big) = (\mu - r)\ dt + \sigma\ dW_t$$

and conclude

$$dS_t = \sigma S_t\ d\tilde{W}_t$$

□

Remark 1.46.
The version of Girsanov's Theorem (see Theorem 1.42) in this thesis does not specify the filtration which underlies the Brownian motion $\tilde{W}_t$. To complete the proof above it is important that $\tilde{W}_t$ is defined on the probability space $(\Omega, \mathcal{F}, (\tilde{\mathcal{F}}_t), \mathbb{Q})$, where $\tilde{\mathcal{F}}_t$ is the natural filtration of W. This can be justified with help of LAMBERTON [9, Theorem 4.2.2], which is a different formulation of Girsanov specifying the filtration. The underlying filtration is important for our use of Martingale Representation Theorem (see Theorem 5.7) in the next proof.

Let us summarize this: If we define $\mathbb{Q}$ as in Girsanov (1.42), we get

$$S_t = \sigma \int_0^t S_u\ d\tilde{W}_u.$$

Therefore, the discounted stock price S is a martingale with respect to the measure $\mathbb{Q}$. Loosely speaking, the risk of S decreasing is the same as the risk of S increasing (with respect to $\mathbb{Q}$), which is the reason why we call $\mathbb{Q}$ the risk-neutral measure. So we answered not only the question of how the risk–neutral measure can be found, but also the question of why we call it risk-neutral. The question of how to use the risk-neutral measure to compute prices for options is still open.

First, we need formal definitions of the terms option and replicable. We find them in Lamberton [9, Page 92]. Again, we focus on European Options:

Definition 1.47.
A european option is a nonnegative random variable $h \in L^2(\tilde{\mathcal{F}}_T, \mathbb{Q})$.

Definition 1.48.
We call a trading strategy (H^0, H) **admissible**, if it is self-financing and if the discounted value $\tilde{V}_t = H_t^0 + H_t S_t$ of the corresponding portfolio satisfies

$$\sup_{t \in [0,T]} \tilde{V}_t \in L^2(\tilde{\mathcal{F}}_T, \mathbb{Q}).$$

Definition 1.49.
We call an option h **replicable** (or **attainable**), if there exists an admissible strategy (H^0, H) such that

$$h = V_T.$$

In this case, the strategy (H^0, H) is called **replicating strategy**, **attaining strategy** or **hedging strategy**.

Loosely speaking, the payoff of the option h has to be equal to the final value of the replicating portfolio.

In the following theorem the process (W_t) is a Brownian motion defined on a probability space, and $(\mathcal{F}_t)$ denotes its natural filtration (see Definition 5.6).

Theorem 1.50 ([9], Theorem 4.3.2, Page 92).
In the Black–Scholes model, every option $h = F_T \in L^2(\mathbb{Q})$ is replicable and the value process of the replicating portfolio satisfies

$$V_t = \mathbb{E}_{\mathbb{Q}}\big(e^{-r(T-t)} \cdot h \mid \mathcal{F}_t\big).$$

Proof.
First assume that there is an admissible strategy (H^0, H) replicating h. The value of

the replicating portfolio is

$$V_t = H_t^0 P_t^0 + H_t P_t,$$

and the discounted value is given by

$$\tilde{V}_t = H_t^0 + H_t S_t.$$

By definition 1.49 we have $h = V_T$.

As our strategy is self-financing, lemma 1.44 yields $d\tilde{V}_t = H_t S_t$, and Lemma 1.45 tells us $dS_t = \sigma \cdot S_t \, d\tilde{W}_t$:

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t H_u \, dS_u = \tilde{V}_0 + \int_0^t H_u \sigma S_u \, d\tilde{W}_u.$$

By our assumptions on H (see Definition 1.48), the discounted value process $\tilde{V}_t$ is a martingale relative to the filtration $(\tilde{\mathcal{F}}_t)$ and the measure $\mathbb{Q}$ (which is the underlying model for the Brownian motion $(\tilde{W}_t)$. Therefore, $\tilde{V}_t$ satisfies for all $0 \leq s \leq t$:

$$\mathbb{E}_{\mathbb{Q}}(\tilde{V}_t \mid \tilde{\mathcal{F}}_s) = \tilde{V}_s,$$

hence

$$\mathbb{E}_{\mathbb{Q}}(\tilde{V}_T \mid \tilde{\mathcal{F}}_t) = \tilde{V}_t$$

and

$$V_t = e^{rt}\tilde{V}_t = e^{rt} \cdot \mathbb{E}_{\mathbb{Q}}(e^{-rT} \underbrace{V_T}_{h} \mid \tilde{\mathcal{F}}_t) = \mathbb{E}_{\mathbb{Q}}(e^{-r(T-t)} \cdot h \mid \tilde{\mathcal{F}}_t)$$

Our goal now is to construct an admissible strategy (H^0, H) which replicates h. We just showed that H^0 and H need to satisfy

$$H_t^0 P_t^0 + H_t P_t = V_t = \mathbb{E}_{\mathbb{Q}}\big(e^{-r(T-t)} h \mid \mathcal{F}_t\big).$$

Note that $M_t := \mathbb{E}_{\mathbb{Q}}\big(e^{-rT} h \mid \mathcal{F}_t\big)$ is a martingale by iterated conditioning (see Example 1.7), which satisfies $M_t \in L^2$ by definition of h (see Definition 1.47). Therefore, by martingale representation theorem (see Theorem 5.7), we can find a previsible process (K_t) such that

$$\underbrace{\mathbb{E}_{\mathbb{Q}}[e^{-rT} h \mid \mathcal{F}_t]}_{M_t} = \underbrace{\mathbb{E}_{\mathbb{Q}}[e^{-rT} h]}_{M_0} + \int_0^t K_s \, d\tilde{W}_t,$$

where we choose $\tilde{W}_t = W_t + \frac{\mu - r}{\sigma} \cdot t$ as in 1.45. Note that this equation implies

$$dM_t = K_t \, d\tilde{W}_t.$$

Now define

$$H_t := \frac{K_t}{\sigma \cdot S_t} \qquad \text{and} \qquad H_t^0 := M_t - H_t S_t.$$

Then we get $M_t = H_t^0 + H_t S_t$ (which tells us M is the discounted value of the portfolio relative to this strategy) and

$$dM_t = K_t \, d\tilde{W}_t = H_t \cdot \sigma \cdot S_t d\tilde{W}_t \overset{1.45}{=} H_t \, dS_t$$

So by 1.44 the strategy (H^0, H) is self-financing with discounted value $\tilde{V}_t = M_t \in L^2(\tilde{\mathcal{F}}_T, \mathbb{Q})$, and $\sup_{t\in[0,T]} \tilde{V}_t = \sup_{t\in[0,T]} \mathbb{E}_{\mathbb{Q}}\big(e^{-rT} h \mid \mathcal{F}_t\big)$ is square integrable (see Remark 1.51), hence (H^0, H) is admissible.

□

Remark 1.51.
Note that to finish the proof of 1.50 we need that $\sup_{t\in[0,T]} \tilde{V}_t = \sup_{t\in[0,T]} \mathbb{E}_{\mathbb{Q}}\big(e^{-rT} h \mid \mathcal{F}_t\big)$ is square integrable. This can easily be shown with help of Doob's L^p-inequality (see Theorem 5.10). To apply this result, one needs to remember that by example 1.7 we know that $\Big(\mathbb{E}_{\mathbb{Q}}\big(e^{-rT} h \mid \mathcal{F}_t\big)\Big)_{t\geq 0}$ defines a martingale.

Remark 1.52.
In the proof of 1.50 we showed that the option h already determines the value process (V_t) of any self-financing strategy (H^0, H). So given a strategy (H_t) of the risky asset we can always define $H_t^0 := V_t - H_t P_t$ to get a self-financing strategy (H^0, H).

1.6.5 Arbitrage

Last but not least we introduce a formal definition of arbitrage (which we already motivated in section 1.6.2) and formulate the Fundamental Theorems of Asset Pricing (FTAP). We use Shreve [17, Pages 230-232] to do so.

In this section we always work with the market model $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, (S_t))$.

Definition 1.53.
We call a portfolio with value process $V(t)$ an **arbitrage opportunity**, if it satisfies $V(0) = 0$ and for some time $T > 0$ we have

$$\mathbb{P}(V(T) \geq 0) = 1$$
$$\mathbb{P}(V(T) > 0) > 0.$$

If there exists an arbitrage opportunity, we say that the market model $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, (S_t))$ admits arbitrage.

So an arbitrage opportunity can be interpreted as a strategy starting with zero capital where one does not lose money almost surely (=no risk) and still has positive probability of making profit.

Theorem 1.54 (FTAP1).
If a market model has an equivalent martingale measure, then it does not admit arbitrage.

Note that the existence of the risk–neutral measure $\mathbb{Q}$ implied by Girsanov in the Black-Scholes Model immediately implies that the Black–Scholes Model does not admit arbitrage. Recall the definition of completeness on page 32.

Theorem 1.55 (FTAP2).
Consider a market model that has an equivalent martingale measure. The market model is complete if and only if this equivalent martingale measure is unique.

Note that in Theorem 1.50 we showed that in the Black–Scholes Model every option is replicable, and therefore the Black-Scholes Model is complete. From FTAP 2 we can deduce that the risk–neutral measure $\mathbb{Q}$ has to be the only equivalent martingale measure in the Black–Scholes Model.

The reader may read Shreve [17, Pages 230-232] for the proofs. A detailed set up of the theory of arbitrage can be found in Delbaen & Schachermayer [4], who also provide lots of formulations of the FTAP's and explain them properly.

In this thesis (and in general literature) it is always reasonable to assume a market model does not admit arbitrage. We will use this later to determine bounds for certain options, when we show that any lower price would lead to the existence of an arbitrage opportunity. We are going to call this assumption the **no-arbitrage-condition**. .

This condition determines the price for any replicable option: If the price differs from the price of the replicating portfolio (i.e V_0), there is arbitrage (for example see Section 1.6.2).

From Theorem 1.50 we directly deduce:

Corollary 1.56.

In the Black–Scholes Model, the price of an option h is given by

$$\mathbb{E}_{\mathbb{Q}}(e^{-r \cdot T} \cdot h).$$

The procedure of generating an arbitrage strategy if the option has the *wrong* price has already been introduced in section 1.6.2.

1.6.6 The Classical Approach: A Short Summary

The procedure from the previous section does not only work in the Black-Scholes model, but in every complete (see Page 32) market. We use HOBSON [7, Section 2.1] for a short summary.
First we postulate a model (maybe depending on parameters) for the stock price process $(P_t)_{t \geq 0}$ defined on some filtered probability space. Then we compute the price of the option F_T as the discounted expectation (with respect to the unique risk–neutral measure $\mathbb{Q}$) of the final outcome:

$$\mathcal{C} = \mathbb{E}_{\mathbb{Q}}\left[e^{-rT} F_T\right]$$

If the price $\mathcal{C}$ depends on parameters such as the strike price and the maturity (which is the case for put or call), we mostly use the notation $\mathcal{C}(K, T)$. In the Black-Scholes model, where drift μ, volatility σ, interest rate r and start price P_0 also play a role we use

$$\mathcal{C}(K, T, P_0, r, \mu, \sigma, r)$$

If the market is complete, this pricing procedure can be justified by

$$e^{-rT} F_T = \mathbb{E}_{\mathbb{Q}}\left[e^{-rT} F_T\right] \; + \; \int_0^T H_t \; dS_t,$$

where H is a self-financing hedging strategy.

2 Hedging and Pricing without a concrete Stochastic Model

2.1 A Motivating Example

In the following section we will try to get information about the price of an option without using any information about an underlying stochastic model. We use HOBSON [7, Chapter 1] as a guideline.

Our goal is to find a model-independent bound for the price of the **digital** option with maturity T:

$$F = \begin{cases} 1 & \max_{0 \leq t \leq T} P_t \geq B \\ 0 & \text{otherwise} \end{cases},$$

where $B \in \mathbb{R}$ is called **Barrier**.

The owner of this option receives a payment of 1 if the (undiscounted) stock price P reaches the barrier B before maturity time T. Of course this option only makes sense if we assume that the initial stock price P_0 is smaller then the barrier B.

Note that we can – instead of this notation above with two cases – also use the notation

$$\begin{aligned} F &= \mathbb{1}_{\{H_B \leq T\}} \\ H_B &= \inf\{u > 0 \mid P_u \geq B\}. \end{aligned}$$

The key observation to reach our goal in this section is the following Lemma:

Lemma 2.1.
For all $K < B$ holds:

$$\mathbb{1}_{\{H_B \leq T\}} \leq \frac{(P_T - K)^+}{B - K} + \frac{(P_{H_B} - P_T)}{B - K} \cdot \mathbb{1}_{\{H_B \leq T\}}$$

Proof.
If the price (P_t) does not hit the barrier before time T, the inequality is trivially true as the left side is zero in that case and on the right side only the nonnegative term is remaining. If the price does hit the barrier, we get

$$1 \leq \frac{(P_T - K)^+ \ + \ B - P_T}{B - K}$$

If $P_T > K$ we have

$$\frac{(P_T - K)^+ \; + \; B - P_T}{B - K} \geq \frac{P_T - K \; + \; B - P_T}{B - K} = 1,$$

and if $P_T < K$ we have

$$\frac{(P_T - K)^+ \; + \; B - P_T}{B - K} = \frac{B - P_T}{B - K} \geq \frac{B - K}{B - K} = 1.$$

$\square$

This inequality shows us that the payoff of our digital option is always smaller or equal then the sum of the payoffs of...

- ...$\frac{1}{B-K}$ call options with strike K and...
- ...$\frac{1}{B-K}$ of a more complicated exotic option where the owner gets the difference of the barrier and the *final stock price* at time T (which can be negative).

We can use the no-arbitrage-condition (see Page 44) to show that this super-replicating strategy determines an upper bound for the price $\mathcal{P}(F)$ of our digital option for any choice of $K < B$:

Theorem 2.2.
Let $C(K)$ denote the price of the call option $(P_T - K)^+$ and let $K < B$. Then the price $\mathcal{P}(F)$ of our digital option satisfies

$$\mathcal{P}(F) \leq \inf_{K<B} \frac{C(K)}{B - K} := \overline{D}.$$

Proof.
Suppose the statement is not true. So there exists $K < B$ such that $\mathcal{P}(F) > \frac{C(K)}{B-K}$. Then with the following strategy we could generate unlimited profit:

1. Let $L > 0$. First, we buy L times the option $(P_{H_B} - P_T) \cdot F$. Note that the owner of this option only benefits if the price process P hits the barrier before time T, and decreases again such that $P_{H_B} > P_T$. Therefore, this option has negative expectation (w.r.t any measure under which P is a martingale), hence the price is 0.
2. Then we buy L times the call option $(P_T - K)^+$, which costs us $L \cdot C(K)$.
3. We sell $L \cdot (B - K)$ times the option F at a (total) price $L \cdot (B - K)\mathcal{P}(F) > L \cdot C(K)$.

4. By our original assumption we made more money selling F that we spent on buying the call option. In the case $H_B \leq T$ we have to pay $L \cdot (B - K)$ to the buyer of F (from step 3).

5. The options we own will get us more money than what we will have to pay the buyer from step 3 by Lemma 2.1:

$$\underbrace{L \cdot (B - K) \cdot F}_{\text{what the buyer gets from us}} \leq \underbrace{L \cdot (P_T - K)^+ + L \cdot (P_{H_B} - P_T) \cdot F}_{\text{what we own}}$$

After all, we made profit without taking any risk, and by increasing L we can increase the profit. Therefore, the assumption we made in the beginning leads to arbitrage, and the statement from the theorem has to hold.

$\square$

Note that to derive this bound we did not make any assumptions on the stochastic model. Nevertheless, we successfully showed that under any riskless measure the discounted expected payoff of the option will be less than $\overline{D}$. This is true because once we fixed such a measure, the price of F is the expected outcome under this measure.

Readers who are interested in why this bound can not be refined and therefore this is optimal are welcome to read chapter 2.7 of Hobson [7].

2.2 Breeden and Litzenberger

In this section we assume that we know prices $C(K)$ of liquidly (=very often, on a regular basis) traded call options for any strike price K and for a fixed maturity T. Hence we can define a function $K \mapsto C(K)$ assigning every strike price K the corresponding call price $C(K)$. Note that since maturity T is fixed there is no need to treat C as a function depending on T and write $C(K, T)$.
The main reference for this section is HOBSON [7, Chapter 2.3]. From there we take that the no-arbitrage condition forces our function C to be decreasing and convex (as a function of the strike price K).

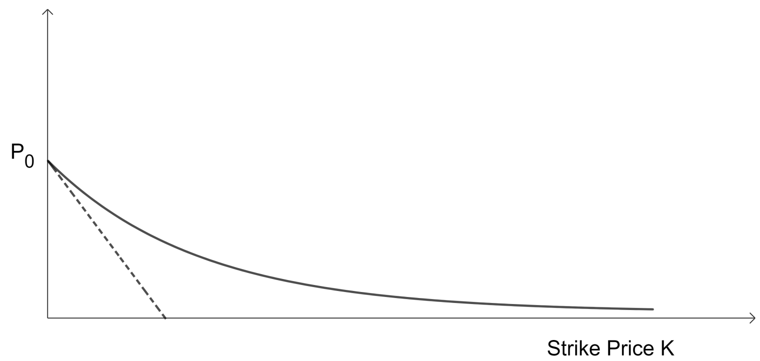


Figure 1: This figure shows a possible graph of C and compares it with the (dashed) graph of $K \mapsto (P_0 - K \cdot e^{-rT})^+$.

The theorem of Breeden and Litzenberger shows how we can use a function like this to construct a measure $\mathbb{Q}$, i.e to construct a stochastic model for our stock price just by using the prices of call option we can observe.

We formulate and prove the following theorem with stronger assumptions than HOBSON [7] does. Readers who are interested in the stronger result and its proof may read the original paper BREEDEN & LITZENBERGER [2].

Theorem 2.3 (Breeden and Litzenberger, see [7], Lemma 2.2).
Fix a maturity $T \in (0, \infty)$ and suppose we know call prices for all strikes K, i.e we can assigne the price of the call option $C(K)$ to every strike price K with a function $K \mapsto C(K)$. Suppose the function $C(K)$ is two times continuously differentiable. Then, assuming the call prices $C(K)$ are calculated as the discounted payoff under a measure $\mathbb{Q}$, i.e

$$C(K) = \mathbb{E}_{\mathbb{Q}}\big[e^{-rT} \cdot (P_T - K)^+\big],$$

we have that

$$\mathbb{Q}(P_T > K) = -e^{rT} C'(K)$$

and

$$\mathbb{Q}(P_T \in dK) = e^{rT} \cdot C''(K).$$

Here, dK means an infinetesimal small intervall on $\mathbb{R}$. This notation becomes clear in the proof. We will prove this result under the assumption that P_T has a density

$$f(K) := \frac{d}{dK}\mathbb{Q}(P_T \leq K),$$

which is continuous but does not have to be differentiable again.

Loosely speaking, the theorem of Breeden–Litzenberger gives us a possibility to construct a stochastic model (i.e a measure $\mathbb{Q}$) under which the observed call prices $C(K)$ occur.

In the proof we use some theory of sequences of functions (see Theorem 5.9) and Leibniz' rule for differentiating integrals (see Theorem 5.8), which is needed because the boundary of the integral depends on the differentiating variable K. Both can be found in the appendix.

Proof.
So let f be the density of P_T under $\mathbb{Q}$. Then:

$$C(K) \;= \mathbb{E}_{\mathbb{Q}}\big[e^{-rT} \cdot (P_T - K)^+\big] \;= e^{-rT} \cdot \int_{\mathbb{R}} (x-K)^+ \cdot f(x) dx \;= e^{-rT} \cdot \int_K^\infty (x-K) \cdot f(x) dx$$

To differentiate this term, define

$$\begin{aligned} H_n(K) &:= \int_K^n (x-K) \cdot f(x)\; dx \\ H(K) &:= \int_K^\infty (x-K) \cdot \; f(x) dx \end{aligned}$$

The sequence H_n converges pointwise against H, because for fixed K and for every $\varepsilon > 0$ we can choose n large enough such that

$$\big|H(K) - H_n(K)\big| = \left|\int_n^\infty (x-K) \cdot f(x)\; dx\right| < \varepsilon$$

To justify this, it is necessary to observe that for $n > K$ we have that

$$\int_n^\infty \underbrace{(x-K) \cdot f(x)}_{\geq 0}\; dx \leq \int_K^\infty \underbrace{(x-K) \cdot f(x)}_{\geq 0}\; dx = C(K) < \infty$$

The function $H_n(K)$ is differentiable by Leibniz' Rule (see Theorem 5.8), and as f is a continuous density function, the derivative is continuous (w.r.t K):

$$\frac{d}{dK}\int_K^n (x-K)\cdot f(x)\ dx = -\int_K^n f(x)\ dx$$

Last but not least we need to observe that the sequence $H_n'(K) = -\int_K^n f(x)\ dx$ converges uniformly to $-\int_K^\infty f(x)\ dx$. This is true because for any $\varepsilon > 0$ we can choose n large enough such that

$$\left|H_n'(K) + \int_K^\infty f(x)\ dx\right| = \left|-\int_K^n f(x)dx + \int_K^\infty f(x)\ dx\right| = \left|\int_n^\infty f(x)\ dx\right| < \varepsilon,$$

and the choice of n can be made independently of K. Then by 5.9 we have that $H(K)$ is differentiable and

$$H'(K) = -\int_K^\infty f(x)\ dx.$$

In total we have

$$C'(K) = e^{-rT}H'(K) = -e^{-rT}\int_K^\infty f(x)\ dx = -e^{-rT}\mathbb{Q}(P_T > K).$$

Now let us turn to the second claim. Instead of dK we work with the interval $[K, K+\delta]$, where we choose $\delta > 0$ to be small:

$$\begin{aligned}\mathbb{Q}(P_T \in [K, K+\delta]) &= \mathbb{Q}(P_T > K) - \mathbb{Q}(P_T > K+\delta) = e^{rT}[C'(K)+\delta) - C(K)] \\ &= e^{rT}\cdot\int_K^{K+\delta} C''(z)dz\end{aligned}$$

This equality is a good way to understand the second claim of the theorem, and in the limiting case $\delta \to 0$ we get

$$\mathbb{Q}(P_T \in dK) = e^{rT}\cdot C''(K)$$

$\square$

Remark 2.4.
Note that lemma 5.9 works only for functions defined on a closed interval $[a, b]$, and we apply it to functions $H_n(K)$ defined on $[0, \infty]$. So for our proof to work, we have to restrict ourselves to the case where we have a maximal strike price K_0. This seems like a hard restriction in theory, but it definitely is no problem in reality.

2.3 Candidate Price Processes

In the last chapter we already saw a possibility to construct the distribution of P_T just by using call prices. But we are still far away from a rigorous stochastic model. Our *main problem* is that, even though we know the distribution of the (undiscounted) price P at time T, we have no idea how the stock prices behaves at other times.

Let us rephrase this: Once we have a distribution for P_T, we can easily determine a distribution μ for the discounted price S_T. The process (S_t), which needs to be a martingale in any market model. But there could be many martingales satisfying $M_T \sim \mu$.

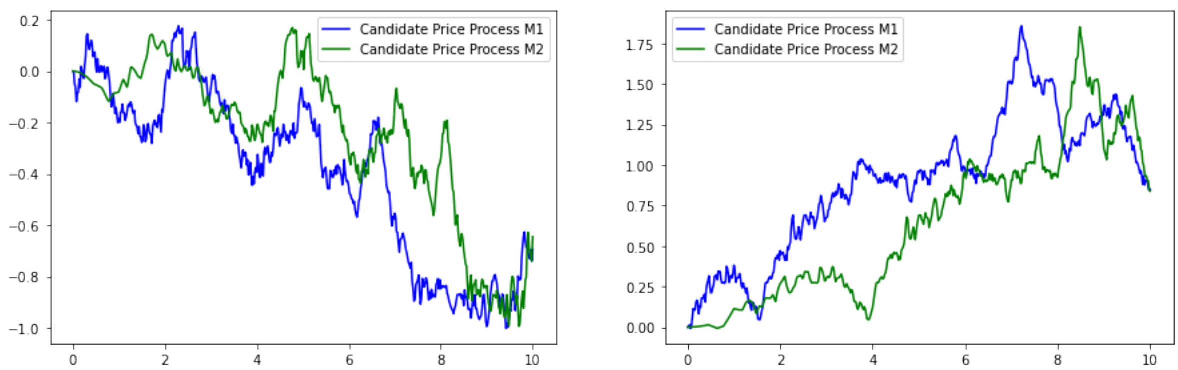
Given a distribution μ on $\mathbb{R}$ and a maturity time T we call a martingale M such that $M_T \sim \mu$ a **candidate price process** (see HOBSON [7, Page 1]).

Example 2.5.
Let us fix $\mu \sim \mathcal{N}(0, \sigma)$ and let $(W_t)_{t\geq 0}$ be a Brownian motion. Since $W_t \sim \mathcal{N}(0, t)$ we scale by $\sqrt{\frac{\sigma}{T}}$ to get distribution μ at time T. Therefore, $M_t^1 := \sqrt{\frac{\sigma}{T}} \cdot W_t$ is a candidate price process. To construct a second process, let $h : [0, T] \to [0, T]$, $h(t) = \frac{t^2}{T}$. Note that h is strictly increasing and satisfies $h(T) = T$. We can observe that $M_t^2 := \sqrt{\frac{\sigma}{T}} \cdot W_{h(t)}$ is a martingale with respect to the filtration $(\mathcal{F}_{h(t)})_{t\geq 0}$:

$$\mathbb{E}(M_t^2 \mid \mathcal{F}_{h(s)}) = \mathbb{E}(\sqrt{\frac{\sigma}{T}} \cdot W_{h(t)} \mid \mathcal{F}_{h(s)}) = \sqrt{\frac{\sigma}{T}} \cdot W_{h(s)} = M_s^2$$

Further, note that $M_T^2 = \sqrt{\frac{\sigma}{T}} \cdot W_{h(T)} = \sqrt{\frac{\sigma}{T}} \cdot W_T \sim \mu$.

In the following figure you can see two simulations of both candidate price processes for maturity time T and variance $\sigma = 0.5$. Note that both processes coincide on time T by construction. Moreover, you can observe that process M^2 fluctuates less at the beginning, as $h(t)$ grows less slowly than identity for small t.



So by using tricks from stochastic analysis we were able to find a nontrivial candidate price process for a fixed probability measure. But how does this work in general? How can we find interesting candidate price processes?

To investigate these questions, and to develop some theory to describe candidate price processes, we will use the connection to the *Skorokhod Embedding Problem* (SEP). As this is the main component of this thesis, we will have a whole chapter to discuss how the theory behind the SEP relates to candidate price processes, and to look at some approaches to solve this problem.

3 The Skorokhod embedding problem

The Skorokhold embedding problem (SEP) as introduced in Hobson [7, Chapter 3] is: Given a stochastic process $(X_t)_{t\geq 0}$ with state space I on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, and a measure μ, find a stopping time τ such that the law of X_τ is μ. We call the problem in this setting the SEP for (X, μ) and we will call τ a solution.

In this thesis, we will (like Hobson [7] does as well) restrict ourselves to the case where $X = W$ is a Brownian motion on $\mathbb{R}$ and μ is a centered probability measure on $\mathbb{R}$, i.e

$$\int_{\mathbb{R}} x \; d\mu(x) = 0$$

We will refer to this restricted setting as the classical version of the SEP. Note that the SEP for (W, μ) has only non-integrable solutions anyway if μ is not centered, as Wald (see Lemma 1.12) states that for any integrable stopping time τ we would have $\mathbb{E}(W_\tau) = 0$.

Let us take a look at example from Mörters & Peres [11, Section 5.3]:

Example 3.1.
Let (W_t) be a Brownian motion and let $-a < 0 < b$. Let μ be such that $\mu(\{-a\}) = \frac{b}{a+b}$ and $\mu(\{b\}) = \frac{a}{a+b}$. We showed in 1.10 that if you choose $\tau_a := \inf\{t > 0 \mid W_t = -a\}$, $\tau_b := \inf\{t > 0 \mid W_t = -b\}$ and $T := \tau_a \wedge \tau_b$ we have

$$W_T \sim \mu.$$

Hence T is a solution for the SEP for (W, μ) and, as we have $\mathbb{E}(T) = a \cdot b$ (see Example 1.11), it is an integrable solution.

It makes sense to look for the first time the Brownian motion has marginal distribution μ:

Definition 3.2.
Let σ and τ be two stopping times for the stochastic process X. We call τ minimal, if $\sigma \leq \tau$ and $X_\sigma \sim X_\tau$ implies $\sigma = \tau$ a.s.

If μ is centered and τ is a minimal solution for the SEP for (W, μ), we write $\tau \in \mathcal{T}(\mu)$.

3.1 The SEP and Candidate Price Processes

In this section we explain how the SEP relates to the pricing problem we discussed so far. But first, let's do a quick recapitulation:

Assume we know all call prices for a fixed maturity T. Then by the Theorem of Breeden and Litzenberger (2.3), we can find find the distribution of the *final stock price* P_T. But as we showed in the last section, it is still unclear how the stock price (P_t) behaves between starting time 0 and maturity time T.

The problem was, given the desired distribution μ of S_T, there might be many martingales (M_t) satisfying $M_T \sim \mu$.

The key result to characterize this class of martingales is the theorem of Dambis-Dubin-Schwarz (see Revuz & Yor [13, Chapter V, Theorem 1.6, Page 181]):

Theorem 3.3 (Dambis-Dubin-Schwarz).
Let M be a continuous local martingale with respect to the filtration $\mathcal{F}_t$, vanishing at 0 such that $[M]_\infty = \infty$, and define

$$\tau_s := \inf\{t > 0 : [M]_t > s\}.$$

Then the process $W_s := M_{\tau_s}$ is an $\mathcal{F}_{\tau_s}$ -Brownian motion and

$$M_t = W_{[M]t}$$

Loosely speaking, the theorem of Dambis-Dubin-Schwarz tells us that martingales satisfying certain assumptions become a Brownian motion after a time change. So let us use this theory to develop an approach to find candidate price processes. We will use the idea of Hobson [7, Section 3.6]. In the following, we will always assume that μ is centered.

Let (M_t) be a continuous martingale with $M_0 = 0$ such that $M_T \sim \mu$. Then by Dambis-Dubin-Schwarz we can write (M_t) as time-changed Brownian motion

$$M_t := W_{[M]_t},$$

and $\tau := [M]_T$ is a solution for the SEP for (W, μ) because $W_{[M]_T} = M_T \sim \mu$ by construction.

On the other hand, if $\tau \in \mathcal{T}(\mu)$, then

$$M_t := W_{\frac{t}{T-t} \wedge \tau}$$

is a martingale with $M_T \sim \mu$. To see this, note that $\frac{t}{T-t} \to \infty$ for $t \to T$ such that $\tau \leq \frac{t}{T-t}$ eventually. To show that M indeed is a martingale, one can use the same technique as in Example 2.5 with $h(t) = \frac{t}{T-t}$.

After all we can obvserve that there is a 1-1 correspondence between candidate price processes and solutions of the SEP, and therefore it is useful to investigate the SEP in order to solve our pricing problem.

3.2 Doob's Approach

In this section we follow HOBSON [7, Section 3.2] to understand Doob's approach for solving the classical version of the SEP.

So let (W_t) be a Brownian motion, and let μ be a centered probability measure on $\mathbb{R}$. We want to find a stopping time τ such that $W_\tau \sim \mu$.

Let F_μ be the distribution function of μ, i.e $F_\mu : \; x \mapsto \mu((-\infty, x])$. Note that $W_1 \sim \mathcal{N}(0,1)$ has standard normal distribution. Let Φ be the distribution function, i.e $\Phi : x \mapsto \mathbb{P}(W_1 \leq x)$.

We denote the generalized inverse of F_μ by F_μ^{-1}, i.e

$$F_\mu^{-1}(y) := \inf\{x \in \mathbb{R} \mid F_\mu(x) \geq y\}$$

Define $Z := F_\mu^{-1}(\Phi(W_1))$. Then $Z \sim \mu$:

$$\begin{aligned}\mathbb{P}(Z \leq x) &= \mathbb{P}(F_\mu^{-1}(\Phi(W_1)) \leq x) = \mathbb{P}(\Phi(W_1) \leq F_\mu(x)) \\ &= \mathbb{P}(W_1 \leq \Phi^{-1}(F_\mu(x))) = \Phi(\Phi^{-1}(F_\mu(x))) = F_\mu(x) = \mu((-\infty, x])\end{aligned}$$

Loosely speaking, we just need to wait until (W_t) hits the value $F_\mu^{-1}(\Phi(W_1)) = Z$, i.e

$$\tau := \inf\{u \geq 1 \; : \; W_u = F_\mu^{-1}(\Phi(W_1))\},$$

to get $W_\tau = F_\mu^{-1}(\Phi(W_1)) \sim \mu$. Note that at time zero $Z = F_\mu^{-1}(\Phi(W_1))$ is a random variable, and after time 1 we know the value of Z and can treat it like any real number.

So with the information available at time 0, we have that $W_\tau \sim \mu$ and therefore (as μ is centered) $\mathbb{E}(W_\tau) = 0$. On the other hand, once we know the value of BM at time 1, say $W_1 = x$, we are in a very different situation.

We observe a Brownian motion starting at time 1 and at value x, i.e the process

$$\tilde{W}_t := W_{t+1} - W_1,$$

which we know is a Brownian motion by the Markov Property.

Assume τ is integrable. Then $\mathbb{E}(\tilde{W}_\tau) = 0$ by Wald's Lemma (1.12). So we have $\mathbb{E}(W_{\tau+1}) = x$. Hence for $x \neq 0$, this yields that $\tau + 1$ is not integrable (by applying Wald's Lemma to the original Brownian motion W), which is a contradiction to the initial assumption that τ is integrable.

To summarize this, Wald's Lemma (1.12) tells us that τ can not be an integrable solution to the SEP if $W_1 \neq 0$, which happens almost surely.

3.3 Hall's Approach

Hall's way of solving the SEP is definitly more complicated than Doob's, but gives us in return a genuine solution which will be integrable. One can find this solution (although not very detailed) in HOBSON [7, Section 3.3].

The general idea is to construct a random interval $[u, v]$, where $u < 0$, and stop when Brownian motion leaves this interval. The key of this solution will be the way we construct the interval.

First, we define

$$c := \int_0^\infty x \; d\mu(x)$$

and note that (because μ is centered) we also have that

$$\int_{-\infty}^0 |x| \; d\mu(x) = -\int_{-\infty}^0 x \; d\mu(x) = c.$$

This can be interpreted as the *weight* the measure μ puts on each side of the vertical axis, i.e and will be important for scaling.

Let $U \in (-\infty, 0)$ and $V \in [0, \infty)$ be two random variables with joint law

$$\rho(du, dv) = \frac{|u| + v}{c} \mu(du)\mu(dv).$$

In a more precise (but less efficient) notation ρ looks like this:

$$\rho([u, u+\delta), [v-\epsilon, v)] = \int_{v-\epsilon}^{v} \int_{u}^{u+\delta} \frac{|x| + y}{c} \; d\mu(x)d\mu(y).$$

Example 3.4.
Suppose μ is a uniform distribution on $(-1, 1)$. For a better intuition, assume X is a random variable with law μ, i.e $X \sim \mu$. Then $F(x) = \frac{1+x}{2}$ is the distribution function, and the density is given by $f(x) = \frac{1}{2}$. Now we compute:

$$c = \int_0^\infty x \; d\mu(x) = \int_0^1 x \; d\mu(x) = \int_0^1 x \cdot \frac{1}{2} \; dx = \frac{1}{4}$$

We used Radon-Nikodym's Theorem the same way as in Example 1.38.

The joint law can now be computed. We assume $-1 < u < 0 < v < 1$:

$$
\begin{aligned}
\rho([u,u+\delta),[v-\epsilon,v)) &= 4\cdot\int_{v-\epsilon}^{v}\int_{u}^{u+\delta}|x|+y\ d\mu(x)d\mu(y)\\
&= 4\cdot\int_{v-\epsilon}^{v}\underbrace{\int_{u}^{u+\delta}|x|\ d\mu(x)}_{-\frac{1}{4}\cdot(\delta^2+2u\delta)}d\mu(y)\ +\ 4\cdot\int_{v-\epsilon}^{v}y\underbrace{\int_{u}^{u+\delta}d\mu(x)}_{\frac{1}{2}\cdot\delta}d\mu(y)\\
&= -(\delta^2+2u\delta)\cdot\delta\cdot u\cdot\underbrace{\int_{v-\epsilon}^{v}d\mu(y)}_{\frac{1}{2}\cdot\epsilon}\ +\ 4\cdot\frac{1}{2}\cdot\delta\cdot\underbrace{\int_{v-\epsilon}^{v}y\ d\mu(y)}_{\frac{1}{4}(2v\epsilon-\epsilon^2)}\\
&= \frac{1}{2}\cdot\epsilon\cdot(-2u\delta-\delta^2)\ +\ \frac{1}{2}\cdot\delta\cdot(2v\epsilon-\epsilon^2)\\
&= \delta\cdot\epsilon\cdot(|u|-\frac{1}{2}\delta\ +\ v-\frac{1}{2}\cdot\epsilon)
\end{aligned}
$$

The integral $\int_u^{u+\delta}|x|\ d\mu(x)$ is easy to compute. We need to choose $\delta>0$ small enough such that $u+\delta$ still is negative:

$$
\int_u^{u+\delta}\underbrace{|x|}_{-x}\ d\mu(x) = -\int_u^{u+\delta}x\cdot\underbrace{\frac{1}{2}}_{\text{density}}\ dx = -\frac{1}{2}\left(\frac{(u+\delta)^2}{2}-\frac{u^2}{2}\right) = -\frac{1}{4}\big(\delta^2+2u\delta\big)
$$

In the notation of HOBSON [7] this reads like

$$
\rho(du,dv) = (|u|+v)\cdot du\cdot dv,
$$

where u and v are the points in the middle of the infinitesimally small intervals du and dv. So ρ is a measure on $[0,1]\times[-1,0]$ which assignes the rectangle $[v-\epsilon,v]\times[u,u+\delta]$ the measure

$$
\delta\cdot\epsilon\cdot\left(|u|-\frac{1}{2}\delta\ +\ v-\frac{1}{2}\cdot\epsilon\right).
$$

Rectangles which have the same size but different position get assigned a different measure. In the figure 2 one can see the rectangle $[v-\epsilon,v]\times[u,u+\delta]$ and compare the two notations. Note that usually by du and dv we mean very small intervals, but for better visibility they are bigger in the picture.

Let us return to the general setting. Our goal now is to show that $\tau=\tau_{U,V}$ is a solution for the SEP, i.e that $W_{\tau_{U,V}}\sim\mu$. Here, as usual, we mean

$$
\tau_{u,v} := \inf\{t>0\mid W_t=u\}\ \wedge\inf\{t>0\mid W_t=v\}.
$$

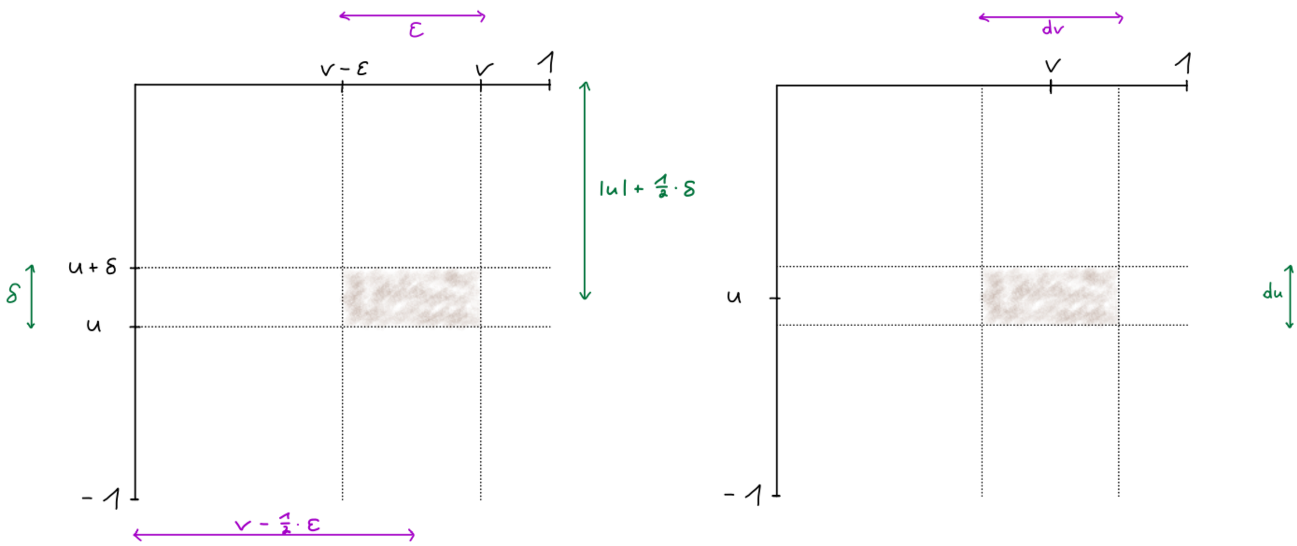


Figure 2: The rectangle $[v - \epsilon, v] \times [u, u + \delta]$ and compare the two notations.

Note that the complication is not the *escape problem* of Brownian motion, which we already solved in the beginning of this thesis (see Example 1.10), but the fact that the boundaries u and v depend on randomness as well, and are not known a priori.

So fix $u_0 < 0$, and let du denote a small intervall containing u_0. We are interested in the probability that Brownian motion stops in this interval. For this we need that the (random) boundary U is in du, and that W hits its lower boundary U first:

$$\mathbb{P}(W_\tau \in du) = \mathbb{P}(U \in du, W_\tau = U)$$

As we don't know the distribution of U, but only the joint law ρ, we have to bring V into our computations. If V could only take finitely many values, we would use the trick

$$\mathbb{P}(U \in du, W_\tau = U) = \sum_v \mathbb{P}(U \in du, W_\tau = U, V = v).$$

As we have $v \in [0, \infty)$, we are dealing with a more delicate situation. For our computation we will use the same idea, but instead of the sum we take an integral:

$$\mathbb{P}(U \in du, W_\tau = U) = \int_{v \in [0,\infty)} \mathbb{P}(U \in du, W_\tau = U, V \in dv)$$

To continue, we'll rewrite the integrand using conditional probability, and use the joint law ρ for further steps. Readers might notice that conditioning on the nullset $\{U =$

$u, V = v\}$ is problematic. However, this issue can be neglected as we already know $\mathbb{P}(W_\tau = u \mid U = u, V = v) = \frac{v}{|u|+v}$.

$$\begin{aligned}
\mathbb{P}\big(W_\tau \in du\big) &= \int_{v\in[0,\infty)} \underbrace{\mathbb{P}\big(U \in du, V \in dv\big)}_{\rho(du,dv)} \cdot \underbrace{\mathbb{P}\big(W_\tau = u|U = u, V = v\big)}_{\frac{v}{|u|+v}} \\
&= \int_{v\in[0,\infty)} \frac{|u|+v}{c}\mu(du)\mu(dv) \cdot \frac{v}{|u|+v} \\
&= \frac{\mu(du)}{c} \cdot \underbrace{\int_{v\in[0,\infty)} v \ d\mu(v)}_{c} \\
&= \mu(du)
\end{aligned}$$

The same computation shows also $\mathbb{P}(W_\tau \in dv) = \mu(dv)$. Therefore $W_\tau \sim \mu$, and τ solves the SEP for (W, μ). But we still don't know whether this is an integrable solution.

Before we start computing $\mathbb{E}(\tau_{U,V})$, remember that in 1.11 we showed that for fixed values $U = u < 0$ and $V = v$ we have $\mathbb{E}(\tau_{u,v}) = |u| \cdot v$. Further, remember that by definition of conditional expectation we have $\mathbb{E}(\tau) = \mathbb{E}\big(\mathbb{E}(\tau_{U,V}|U, V)\big)$.

Lemma 3.5.
If μ is a measure with finite variance (meaning any random variable X with law μ has finite variance), then Hall's solution $\tau = \tau_{U,V}$ is integrable.

Proof.

$$\begin{aligned}
\mathbb{E}(\tau) &= \int_{-\infty}^{0}\int_{0}^{\infty} |u| \cdot v \cdot \rho(du, dv) \\
&= \int_{-\infty}^{0}\int_{0}^{\infty} |u| \cdot v \cdot \frac{|u|+v}{c} \cdot \mu(du)\mu(dv) \\
&= \int_{-\infty}^{0}\int_{0}^{\infty} \frac{|u|^2 \cdot v}{c}\mu(du)\mu(dv) \ + \ \int_{-\infty}^{0}\int_{0}^{\infty} \frac{|u| \cdot v^2}{c}\mu(du)\mu(dv) \\
&= \int_{-\infty}^{0} \frac{|u|^2}{c}\mu(du) \underbrace{\int_{0}^{\infty} v \ d\mu(v)}_{c} \ + \ \underbrace{\int_{-\infty}^{0} |u|\mu(du)}_{c} \int_{0}^{\infty} \frac{v^2}{c} \ d\mu(v) \\
&= \int_{-\infty}^{0} u^2 d\mu(u) \ + \ \int_{0}^{\infty} v^2 d\mu(v) \\
&= \int_{-\infty}^{\infty} x^2 \ d\mu(x) < \infty
\end{aligned}$$

3.4 Properties and Existence of Solutions

In the last sections we saw two ways of solving the SEP. An important difference between them is integrability: We showed that Doob's approach does not result in an integrable solution, and Hall's solution is integrable under the condition that μ has finite variance. Further, we already clarified that we are mainly interested in minimal solutions, i.e stopping times τ such that there exists no stopping time σ with $W_\sigma \sim W_\tau$ and $\sigma \leq \tau$.

This raises questions: Does there always exist a solution satisfying both of these conditions? What other properties can we deduce about integrable solutions?

First of all, it is not difficult to show that all integrable solutions are minimal:

Lemma 3.6.
Let τ be a stopping time such that $\mathbb{E}(\tau) < \infty$. Then $\mathbb{E}(W_\tau) = 0$, $\mathbb{E}(W_\tau^2) = \mathbb{E}(\tau)$ and τ is minimal.

Proof.
First, note that $\mathbb{E}(W_\tau) = 0$ and $\mathbb{E}(W_\tau^2) = \mathbb{E}(\tau)$ are the properties we get from Wald's Lemma (1.12, 1.13). Now assume $\sigma \leq \tau$ and $W_\sigma \sim W_\tau$:

$$\mathbb{E}(\sigma) = \mathbb{E}(W_\sigma^2) = \mathbb{E}(W_\tau^2) = \mathbb{E}(\tau)$$

Therefore, $\sigma = \tau$ a.s and τ indeed is a minimal stopping time.

□

So if we have an integrable solution τ of the SEP for (W, τ), we can conclude that μ has finite variance. To show this, let X be a random variable with law μ. From Wald's Lemma (1.13) we get that $\mathbb{E}(W_\tau^2) = \mathbb{E}(\tau) < \infty$, and because W_τ has law μ we have

$$V(X) = \int_{\mathbb{R}} x^2 \; d\mu(x) = \mathbb{E}(W_\tau^2) = \mathbb{E}(\tau) < \infty.$$

Conversely, we see that if μ does not have finite variance, there can't be an integrable solution. If μ does have finite variance, we can not conclude that any solution τ has to be integrable (see [7, Corollary 3.3, Page 16]). However, Hall's approach makes sure that if μ has finite variance there exists an integrable solution.

Another important property is the following:

Lemma 3.7 ([7], Corollary 3.4).
If μ is centered and has support contained in an interval $[-a, b]$, where $a, b > 0$ and τ is

a minimal solution of the SEP for (W, μ), then τ is smaller or equal than the first exit time $\tau_a \wedge \tau_b$ of the interval $[-a, b]$.

Proof.
As the support of μ is contained in an interval, μ has finite variance. So there exists an integrable solution σ (provided by Hall for instance), and as τ is minimal we have $\tau \leq \sigma$ and therefore $\mathbb{E}(\tau) \leq \mathbb{E}(\sigma) < \infty$.

(Remark: σ is minimal as well as all integrable stopping times are minimal by Lemma 3.6, so $\sigma \leq \tau$, hence $\sigma = \tau$. This observation is not important for the proof.)

Now $W_\tau \in [-a, b]$ a.s, hence

$$0 \leq \mathbb{E}\big(\underbrace{(W_\tau + a)}_{\geq 0} \cdot \underbrace{(b - W_\tau)}_{\geq 0}\big) \overset{\mathbb{E}(W_\tau)=0}{=} ab - \mathbb{E}(W_\tau^2).$$

Recall that $\mathbb{E}(\tau_a \wedge \tau_b) = ab$ (see Example 1.11) and $\mathbb{E}(W_\tau^2) = \mathbb{E}(\tau)$ by Wald (see Lemma 1.13):

$$\mathbb{E}(\tau_a \wedge \tau_b) \overset{1.11}{=} ab \geq \mathbb{E}(W_\tau^2) \overset{1.13}{=} \mathbb{E}(\tau)$$

This inequality contradicts the assumption $\tau > \tau_a \wedge \tau_b$, hence we have

$$\tau_a \wedge \tau_b \leq \tau.$$

□

Remark 3.8.
Let us summarize our observations about the existence of solutions to the SEP.

- In the beginning of chapter 3 we established with help of Wald's Lemma (1.12) that there can not be an integrable solution if μ is not centered, because if we had $\mathbb{E}(\tau) < \infty$, we would also have $\mathbb{E}(W_\tau) = 0$.
- In this section we saw that there is also no integrable solution to the SEP of (W, μ) if μ does not have finite variance.
- If μ is centered with finite variance, there is still the possibility that a solution τ is not integrable, as we saw in Doob's approach.
- If μ is centered and has finite variance, Hall's approach provides us with an integrable solution.

3.5 Root's Approach

We have already seen that we can construct a solution for the SEP by using hitting times. For instance, Hall constructed a random interval (U, V) and defined the desired stopping time as the first time the Brownian motion W hits the boundary of (U, V). Root, likewise to Hall, also follows the concept of stopping when Brownian motion hits some kind of boundary. The difference to Hall's approach is that he focuses not on the one-dimensional process $(W_t)_{t\geq 0}$, but on the two-dimensional process $(t, W_t)_{t\geq 0}$.

In this section we will follow Hobson[7, Chapter 5], the original paper of Root [15] and Cox & Wang [3].

Definition 3.9.
We call a closed subset of $[0, \infty] \times [-\infty, \infty]$ a *Barrier*, if

i) The tuple (∞, t) is element of B for every $x \in [-\infty, \infty]$.

ii) The tuples (t, ∞) and $(t, -\infty)$ are element of B for every $t \in [0, \infty]$.

iii) If $(t, x) \in B$, then for every $s > t$ we have that $(s, x) \in B$.

Root's approach can be summarized as follows: For a given law μ find a barrier B such that the stopping time $\tau_B := \inf\{u \geq 0 \mid (u, W_u) \in B\}$ is a solution of the SEP for (W, μ).

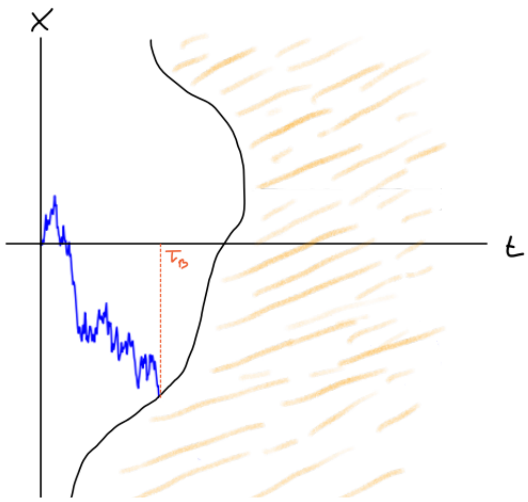


Figure 3: The Root barrier B highlighted in orange stripes, and the Brownian motion hitting the barrier at time τ_B.

Note that condition iii) makes sure that every barrier B can be written in the form $\mathcal{R}_b := \{(t, x) \in [0, \infty] \times [-\infty, \infty] \mid t \geq b(x)\}$ for a function b. This function, however, can also take ∞ as a value, and does not need to be continuous as we will see in example 3.10.

Example 3.10 ([7], Page 32).

i) Suppose $\mu \sim \mathcal{N}(0,1)$. We know $W_1 \sim \mu$, so our barrier is given by the function $b(x) = 1$.

ii) Let $\mu = \frac{1}{2} \cdot (\delta_{-1} + \delta_1)$, i.e $\mu(\{1\}) = \mu(\{-1\}) = \frac{1}{2}$. We already showed the SEP can be solved by the minimal hitting time $\tau_{-1} \wedge \tau_1$. Hence $b(x) = 0$ for $x \leq -1$ or for $x \geq 1$, and $b(x) = \infty$ for $x \in (-1,1)$. The barrier is given by $B = \{(t,x) \mid x \leq -1 \vee x \geq 1\}$. Note that we could choose the barrier B (resp. the barrier function b) differently for $x > 1$ or $x < -1$ since we stop at $x = -1$ or $x = 1$ anyway. Therefore, the barrier is not unique.

iii) Suppose $\mu = p \cdot \delta_{-1} + p \cdot \delta_1 + (1-2p) \cdot \delta_0$, i.e $\mu(\{1\}) = \mu(\{-1\}) = p$ and $\mu(\{0\} = 1-2p$, which is defined for $0 \leq p \leq \frac{1}{2}$. In this case, we need to add a set $\{(t,0) \mid t \geq t_0(p)\}$ to the barrier of case ii) to allow (t, W_t) to stop at value 0 as well. This set has to depend on p:

   If p increases, it becomes less likely for the Brownian motion to stop at 0, and $t_0(p)$ increases as well. In the case $p = 0$ we have $t_0(0) = 0$ because we need Brownian motion to stop immediately (as $\mu = \delta_0$ in that case). Further, we have $t_0(\frac{1}{2}) = \infty$ because $p = \frac{1}{2}$ implies that Brownian motion can not stop at value 0. This value leads us directly to example ii).

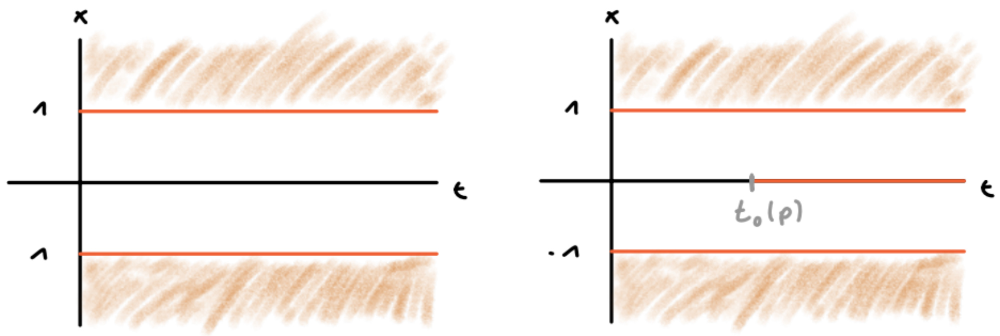


Figure 4: The barrier for example iii) hightlighted in color.

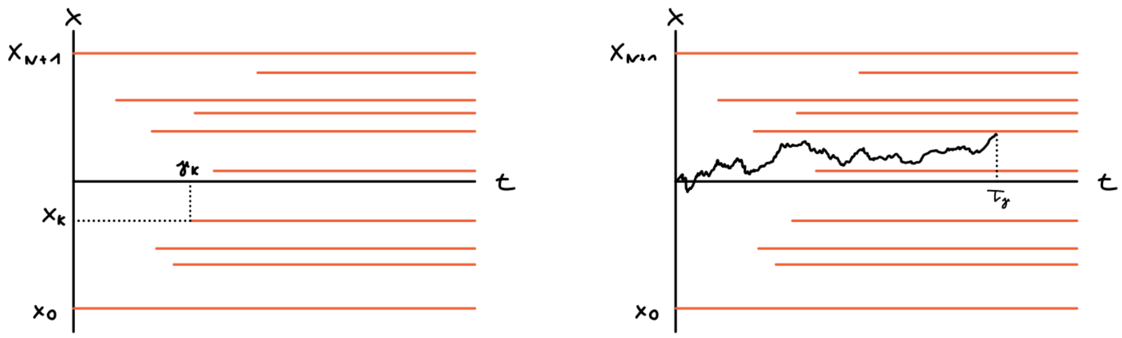
Theorem 3.11 ([15], Page 1 and [7], Page 31).
Let μ be a centered probability measure. There exists a barrier function $b : [-\infty, \infty] \to [0, \infty]$ such that the stopping time $\tau := \inf\{u \geq 0 \mid (u, W_u) \in \mathcal{R}_b\}$ is a solution of the SEP for (W, μ).

We only prove this theorem for the case that μ is an atomic measure, meaning we assume there is a finite set $\mathcal{X} = \{x_0, x_1, \dots, x_N, x_{N+1}\}$ such that $\mu(\{x_i\}) = p_i$ for $i = 0, \dots, N+1$ and $\sum_{i=0}^{N+1} p_i = 1$.

Proof.
The goal of the proof is to show the existence of the barrier function b. Because of the simplifying assumption that μ is atomic, we only need to find the values $b(x_i)$ for $x_i \in \mathcal{X}$. As the Brownian motion can never stop at any other value, we set $b(x) = \infty$ for any $x \notin \mathcal{X}$.

As μ is centered and has support contained in the interval $[x_0, x_{N+1}]$, we already know τ needs to be less or equal to the exit time $\tau_{x_0} \wedge \tau_{x_{N+1}}$, because otherwise τ can not be minimal as we showed in the last section (see Corollary 3.7). To assure this, we set $b(x_0) = b(x_{N+1}) = 0$.

Let $\gamma = (\gamma_1, \dots, \gamma_N)$ be a vector in $\mathbb{R}_+^N$, i.e $\gamma_i \geq 0$. We define $\tau_\gamma := \inf\{u \geq 0 \mid W_u = x_i, u \geq \gamma_i, 0 \leq i \leq N+1\}$.



Our goal is to show that there is a vector γ^* such that $W_{\tau_{\gamma^*}} \sim \mu$, so we can define the barrier-function $b(x_i) = \gamma_i^*$ and the proof is finished.

To achieve this, we define $\Gamma = \{\gamma \in \mathbb{R}_+^N \mid \mathbb{P}(W_{\tau_\gamma} = x_i) \leq p_i = \mu(x_i)\}$ and show that Γ has a minimal element.

Note that for $\gamma \in \Gamma$ we have $\sum_{i=1}^N \mathbb{P}(W_{\tau_\gamma} = x_i) \leq 1 - p_0 - p_{N+1}$, implying that we need $\mathbb{P}(W_{\tau_\gamma} = x_0) + \mathbb{P}(W_{\tau_\gamma} = x_{N+1}) \geq p_0 + p_{N+1}$. This is true because we obviously have $\sum_{i=0}^{N+1} \mathbb{P}(W_{\tau_\gamma} = x_i) = 1$ by definition of τ_γ.

We claim: If $\tilde{\gamma}$ and $\hat{\gamma}$ are elements of Γ, then $\underline{\gamma} \in \Gamma$, where $\underline{\gamma_i} := \tilde{\gamma}_i \wedge \hat{\gamma}_i$.

- Fix $1 \leq i \leq N$, w.l.o.g we assume $\hat{\gamma}_i \leq \tilde{\gamma}_i$.
- Then $\tau_{\underline{\gamma}} = \tau_{\hat{\gamma}}$ on the set $\{W_{\tau_{\underline{\gamma}}} = x_i\}$, and $\tau_{\underline{\gamma}} \leq \tau_{\hat{\gamma}}$ otherwise.
- So we have $\{\omega \in \Omega \ : \ W_{\tau_{\underline{\gamma}}} = x_i\} \subseteq \{\omega \in \Omega \ : \ W_{\tau_{\hat{\gamma}}} = x_i\}$, and therefore

$$\mathbb{P}(W_{\tau_{\underline{\gamma}}} = x_i) \leq \mathbb{P}(W_{\tau_{\hat{\gamma}}} = x_i) \leq p_i$$

as $\hat{\gamma} \in \Gamma$. This proves the claim.

Therefore, there is a minimal element γ^* such that $\gamma^* \wedge \gamma = \gamma^*$ for all $\gamma \in \Gamma$. We claim that γ^* emmbeds μ, i.e $W_{\tau_{\gamma^*}} \sim \mu$.

- Assume γ^* does not embed μ. Then there exists $1 \leq i \leq N$ sucht that $\mathbb{P}(W_{\tau_{\gamma^*}} = x_i) < p_i$.
- So we can reduce γ_i^* without hurting the condition $\mathbb{P}(W_{\tau_{\gamma^*}} = x_i) < p_i$, which makes it more likely that Brownian motion stops at value x_i.
- As this only reduces the probabilities $\mathbb{P}(W_{\tau_{\gamma^*}} = x_j)$ for $j \neq i$ without changing γ_j, the reduced vector is still in Γ. Hence γ^* is not minimal. A contradiction.

This finishes the proof for the atomic measure μ. The general result follows by extending this simplyfied result with theory from the field of topology. Interested readers find this in Rost [16].

$\square$

3.6 Optimality of Root's Solution

The definition of optimality in this context comes from ROST [16]. He called a solution τ of the SEP for (W, μ) **optimal**, if for any $t \in \mathbb{R}^+$ it minimizes **residual expectation** $\mathbb{E}((\tau - t)^+)$, i.e for any $t \in \mathbb{R}^+$ and for any other solution $\tilde{\tau}$ of the SEP for (W, μ) a solution τ is called optimal if it satisfies

$$\mathbb{E}((\tau - t)^+) \leq \mathbb{E}((\tilde{\tau} - t)^+).$$

In this section we will show that the root solution τ_B is optimal in the sense of ROST [16] using mostly COX & WANG [3, Chapter 5]. First, we need to observe the following:

Lemma 3.12.
A solution τ of the SEP for (W, μ) is of minimal residual expectation if and only if for every convex, increasing function F with $F(0) = F'_+(0) = 0$ it minimizes the quantity

$$\mathbb{E}(F(\tau))$$

Proof.
By F'_+ we denote the right derivative of F. From now on we will write only write f instead of F'_+. Note that one direction of this equivalence is trivial as the function $F(\tau) := (\tau - t)^+$ is convex, increasing and satisfies $F(0) = f(0) = 0$.

The other direction follows from

$$\int_0^\infty (\tau - t)^+ \, dF''(t) = F(\tau).$$

Note that, as F is convex, the function f is nondecreasing and therefore the measure $dF''((a, b]) := F'_+(b) - F'_+(a) = f(b) - f(a)$ exists. We show this equation simply by using integration:

$$\begin{aligned}\int_0^\infty (\tau - t)^+ \, dF''(t) &= \underbrace{\lim_{L\to\infty} f(L) \cdot (\tau - L)^+}_{0} - \underbrace{f(0)}_{0} \cdot (\tau - 0)^+ - \int_0^\infty f(t) \; \underbrace{d(\tau - t)^+}_{0 \ldots \tau < t} \\ &= \int_0^\tau f(t) \, dt = F(\tau)\end{aligned}$$

Here you need to observe that, as $(\tau - t)$ is decreasing (in t), therefore we technically integrate against a so called *signed measure* $d(\tau - t)^+$. We can easily fix this by changing the sign of the integral and integrating against $d(-(\tau - t)^+)$, which gives us 0 for $t > \tau$.

Further, you might note that $f = F'_+$ does not need to be continuous, hence it is not clear whether we can use $\int_0^\tau f(t) \, dt = F(\tau)$. However, we may use this form of the fundamental theorem because F is convex. One finds this in ROCKAFELLAR[14, pp. 24.2, 24.2.1].

$\square$

To summarize this: Our goal in this section is to show the following:

Suppose τ_B is the Root solution of the SEP for (W, μ), and suppose τ_B is integrable. Then, under the assumption that $f = F'_+$ is bounded, for any other integrable solution τ of the SEP for (W, μ) we have

$$\mathbb{E}(F(\tau_B)) \leq \mathbb{E}(F(\tau)).$$

The precice statement will be formulated later in theorem 3.14, because the reader is not familiar with all the necessary notatation yet.

Note that, different than COX & WANG [3], we only show optimality for integrable solutions. However, this statement is correct if you also consider non integrable solutions of the SEP. Further, one might note that COX & WANG [3] work not only with the classical SEP like we do in this thesis, but show optimality for the more general SEP (X, μ) where X_t satisfies $dX_t = \sigma_t \, dW_t$.

A detailed introduction of this more general setting can be read right at the beginning of chapter 4.

Before we start the proof of the statement above, we need to do some preparations. To avoid confusion, we define $\tau_B^s := \inf\{t \geq s \mid (t, W_t) \in B\}$, i.e the stopping time τ_B from the theorem can be denoted by τ_B^0. Further, we need some additional assumptions for the desired theorem to hold.

Preparation Phase I: The Function M

The first preparation we need is the function

$$M(t, x) := \mathbb{E}(f(\tau_B^t) \mid W_t = x).$$

In this context, three different cases might occur:

In the left picture of figure 3.6 we observe that Brownian motion has not crossed the barrier at time t yet. The function M now tells us what, given the information $W_t = x$, we have to expect for the value $f(\tau_B)$. One needs to be careful because it is easy to confuse $\mathbb{E}(f(\tau_B))$ with $f(\mathbb{E}(\tau_B))$ here.
In the picture in the middle of figure 3.6 you can observe that $(t, x) \in B$. In this case we have $\tau_B^t = \inf\{u \geq t \mid (u, W_u) \in B\} = t$, and therefore $M(t, x) = f(t)$.
The most interesting case is probably the one we can observe in the right picture of figure

3.6. Here, the Brownian motion has crossed the Barrier already, but came back in the compliment B^C again. So $t > \tau_B^0$, but anyway we don't have $\tau_B^t = t$ in this case.

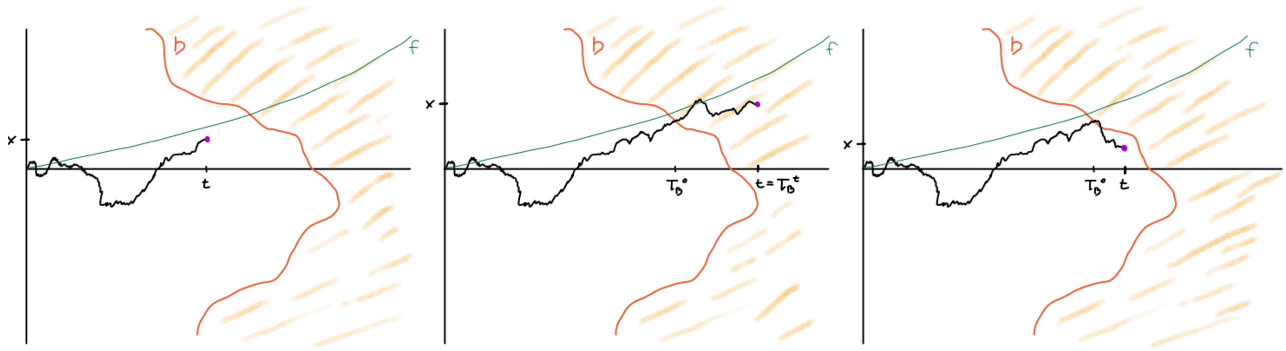


Figure 5: Three important cases for the function M.

Note that neither f, nor b needs to be continuous as they are in the pictures above.

In the left picture we observe that Brownian motion has not crossed the barrier at time t yet. The function M now tells us what, given the information $W_t = x$, we have to expect for the value $f(\tau_B)$. One needs to be careful because it is easy to confuse $\mathbb{E}(f(\tau_B))$ with $f(\mathbb{E}(\tau_B))$ here.
In the picture in the middle you can observe that $(t, x) \in B$. In this case we have $\tau_B^t = \inf\{u \geq t \mid (u, W_u) \in B\} = t$, and therefore $M(t, x) = f(t)$.
The most interesting case is probably the right one. In this picture, the Brownian motion has crossed the Barrier already, but came back in the compliment B^C again. So $t > \tau_B^0$, but anyway we don't have $\tau_B^t = t$ in this case.

Note that the function M always has the property $M(t, x) \geq f(t)$. This is true because f is a nondecreasing function, so at time t we can not expect f to be smaller than the current value at a future time τ_B^t.

Later in the proof we will also be confronted with the term $M(0, W_s)$ for some $s > 0$. This can be thought of a restart of the whole procedure: First, you watch the Brownian motion until time s, then you start again at time 0 and at the value W_s you stopped (see figure 6).

We can even consider $M(s, W_{\tau_B})$, where we restart the whole process at time s. Remember that if $s \geq \tau_B^0$, we restart the process at a point $(s, W_{\tau_B^0}) \in B$ (see Definition 3.9) and get $M(s, W_{\tau_B^0}) = f(s)$.

Figure 7 shows the nontrivial case where the process is not restarted in B.

It is important to mention that we consider a new process after the restart. That is why we can call the first hitting time of the barrier τ_B^0 (respectively τ_B^s) again. The time

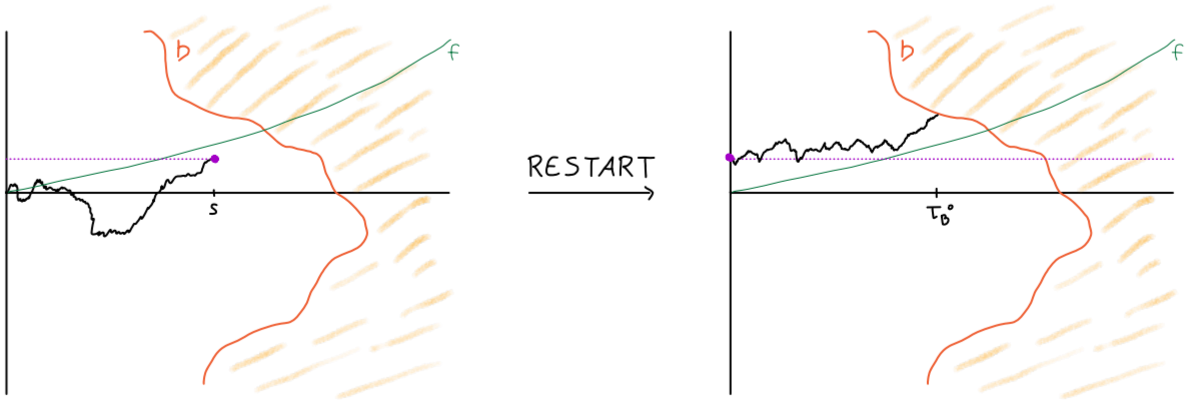


Figure 6: Restart at time zero before hitting the barrier.

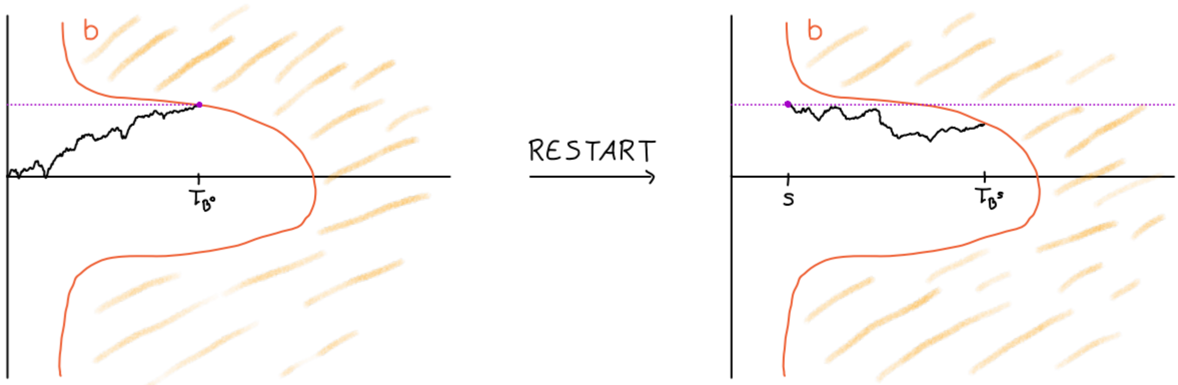


Figure 7: Restart at time s exactly when the barrier is being hit for the first time.

which has passed until the original process reached the *restarting level* does not count here anymore.

Last but not least, note that if one assumes f to be bounded, we trivially get that M is locally bounded.

Preparation Phase II: The Function Z

The second function we need to define is

$$Z(x) = 2 \cdot \int_0^x \int_0^y M(0, z)\ dz\ dy.$$

Note that we have an expectation with the condition $W_0 = z$, which technically is only well defined for $z = 0$ if we restrict ourselves to standard Brownian motion. In the setting of this proof, we need to adapt our concept of Brownian motion a little bit: Instead of strictly starting at value zero, we let the condition $\{W_0 = z\}$ determine the starting point of Brownian motion.

Observe that we can differentiate Z twice, but Z'' might not be continuous:

$$Z'(x) = 2 \cdot \int_0^x M(0,z)dz$$
$$Z''(x) = 2 \cdot M(0,x) > 0$$

However, it is evident that $Z'' > 0$ and therefore Z is convex. This gives us the possibility to use a *special version* of Itô's formula (which does not require Z'' to be continuous) called Meyer-Itô formula. It can be found in PROTTER [12, Theorem IV.71]. We get:

$$Z(W_t) = Z(W_0) + \int_0^t Z'(W_t)dW_t + \frac{1}{2}\int_0^t Z''(W_t)dt$$

This will be useful in the proof later. Last but not least, observe that we always have $Z(x) \geq 0$. This fact seems clear for $x \geq 0$, and for $x < 0$ we have

$$Z(x) = \int_0^x \int_0^y M(0,z)\ dz\ dy \overset{x\leq 0}{=} -\int_x^0 \int_0^y M(0,z)\ dz\ dy \overset{y\leq 0}{=} \int_x^0 \int_y^0 \underbrace{M(0,z)}_{\geq 0}\ dz\ dy \geq 0$$

Preparation Phase III: The Functions G and H

We define

$$G(t,x) := \int_0^t M(s,x)\ ds - Z(x)$$
$$H(x) = \int_0^{b(x)} \big(f(s) - M(s,x)\big)\ ds + Z(x)$$

Observe that for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ we have $G(t,x) + H(x) \leq F(t)$:

- If $t < b(x)$, then

$$G(t,x) + H(x) = \int_0^t M(s,x)\ ds + \int_0^t \big(f(s) - M(s,x)\big)\ ds + \int_t^{b(x)} \underbrace{\big(f(s) - M(s,x)\big)}_{\leq 0}\ ds$$
$$\leq \int_0^t f(s)\ ds = F(t)$$

- If $t \geq b(x)$, then $(t,x) \in B$ and we can even show equality:

$$G(t,x) + H(x) = \int_{b(x)}^t \underbrace{M(s,x)}_{=f(s)}\ ds + \int_0^{b(x)} f(s)\ ds = F(t)$$

Preparation Phase IV: The Key Lemma

Lemma 3.13 ([3], Lemma 5.2).
Suppose f is bounded and for any $T > 0$ we have $\mathbb{E}(Z(W_0)) < \infty$ and

$$\mathbb{E}\left(\int_0^T Z'(W_s)^2 \, ds\right) < \infty.$$

Then the process

$$G(t \wedge \tau_B^0, W_{t\wedge\tau_B^0})$$

is a martingale and the process

$$G(t, W_t)$$

is a submartingale.

One can find the proof of this Lemma in Cox & Wang [3, Page 882].

The Proof of Optimality

We are finally ready to state and prove the desired theorem:

Theorem 3.14 ([3], Theorem 5.3).
Suppose τ_B is the Root solution of the SEP for (W,μ), and suppose τ_B is integrable. Then, under the assumptions that $f = F'_+$ is bounded, $\mathbb{E}(Z(W_0)) < \infty$ and

$$\mathbb{E}\bigg(\int_0^T Z'(W_s)^2 \; ds\bigg) < \infty$$

for any other integrable solution τ of the SEP for (W,μ), we have

$$\mathbb{E}(F(\tau_B)) \leq \mathbb{E}(F(\tau)).$$

Note that the assumption that M is locally bounded (as we find it in [3]) is not necessary, as we make the stronger assumption that f is bounded.

Proof.
First of all, note that we satisfy all assumptions of our key lemma (Lemma 3.13), hence we may use that the process

$$G_t := G(t, W_t) = \int_0^t M(s, W_t) \; ds \; - Z(W_t)$$

is a submartingale. Therefore, the stopped process $(G_{t\wedge\tau})$ is a submartingale as well by OST (see Remark 1.9). Now let τ be any integrable solution of the SEP (W,μ) and let τ_B^0 be Root's solution. We divide the proof in three steps:

Step 1: <u>Show $\mathbb{E}\big(G(t\wedge\tau, W_{t\wedge\tau})\big) \to \mathbb{E}\big(G(\tau, W_\tau)\big)$.</u>

We start our argumentation by establishing almost sure convergence. By Theorem 5.11 it suffices to show that $(G_{t\wedge\tau})$ is bounded in L^1. To do so, remember that we have $Z(x) \geq 0$ for all $x \in \mathbb{R}$ and $f(t) \leq \lim_{s\to\infty} f(s) =: f(\infty)$:

$$G_{t\wedge\tau} = \int_0^{t\wedge\tau} \underbrace{M(s, W_{t\wedge\tau})}_{\leq f(\infty)} \; ds \; - Z(W_{t\wedge\tau}) \leq f(\infty)\cdot\tau$$

As we need to bound the absolute value of $G_{t\wedge\tau}$, we need to find a lower bound as well. Therefore, it is necessary to estimate $Z(W_{t\wedge\tau})$.

We apply Meyer-Itô's formula:

$$\begin{aligned}\mathbb{E}\big(Z(W_{t\wedge\tau})\big) &= \mathbb{E}(Z(W_0)) + \underbrace{\mathbb{E}\Big(\int_0^{t\wedge\tau} Z'(W_s)\; dW_s\Big)}_{=0 \text{ because martingale}} + \frac{1}{2}\cdot\mathbb{E}\Big(\int_0^{t\wedge\tau} \underbrace{Z''(W_s)}_{2\cdot M(0,W_s)\leq 2\cdot f(\infty)}\; ds\Big)\\ &\leq \mathbb{E}(Z(W_0)) + f(\infty)\cdot\mathbb{E}(t\wedge\tau) < \infty\end{aligned}$$

With help of Fatou's Lemma (see Lemma 5.3) and the facts that Z is continuous and τ is finite a.s (as it is integrable) we conclude that $Z(W_\tau)$ is integrable:

$$\mathbb{E}\big(Z(W_\tau)\big) = \mathbb{E}\big(Z(\lim_{t\to\infty} W_{\tau\wedge t})\big) = \mathbb{E}\big(\lim_{t\to\infty} Z(W_{\tau\wedge t})\big) = \mathbb{E}\big(\liminf_{t\to\infty} Z(W_{\tau\wedge t})\big) \leq \liminf_{t\to\infty}\mathbb{E}\big(Z(W_{\tau\wedge t})\big) < \infty$$

As Z is convex and $Z(W_\tau)$ is integrable, we may use Jensen's inequality (see Theorem 5.14) and the OST (see Theorem 1.8) to show $Z(W_{t\wedge\tau}) \leq \mathbb{E}\big(Z(W_\tau) \mid \mathcal{F}_t\big)$:

- If $t \leq \tau$ we have $Z(W_{t\wedge\tau}) = Z(W_t) \overset{OST}{=} Z\big(\mathbb{E}(W_\tau \mid \mathcal{F}_t)\big) \leq \mathbb{E}\big(Z(W_\tau) \mid \mathcal{F}_t\big)$.
- If $t > \tau$ we have $Z(W_{t\wedge\tau}) = Z(W_\tau) = \mathbb{E}\big(Z(W_\tau) \mid \mathcal{F}_t\big)$.

Now we have that

$$G_{t\wedge\tau} = \underbrace{\int_0^{t\wedge\tau} M(s, W_{t\wedge\tau})\; ds}_{\geq 0} - Z(W_{t\wedge\tau}) \geq -Z(W_{t\wedge\tau}) \geq \mathbb{E}\big(Z(W_\tau) \mid \mathcal{F}_t\big)$$

Note that $\mathbb{E}\big(Z(W_\tau) \mid \mathcal{F}_t\big)$ is trivially a closed martingale (see Definition 5.12), and therefore by Theorem 5.13 L^1 convergent, uniformly integrable, and a.s convergent, hence L^1 bounded.

As we succeeded in bounding the submartingale $G_{t\wedge\tau}$ from above and from below by L^1 functions, it is safe to say that $G_{t\wedge\tau}$ is L^1 bounded, hence a.s convergent by Theorem 5.11.

The convergence in L^1 follows directly from dominated convergence theorem (5.1). We can use $|G_{t\wedge\tau}| \leq \max\{\big|\sup_{t\in\mathbb{R}}\mathbb{E}\big(Z(W_\tau) \mid \mathcal{F}_t\big)\big|, \big|f(\infty)\cdot\tau\big|\}$:

$$\lim_{t\to\infty}\mathbb{E}(G_{t\wedge\tau}) \overset{DCT}{=} \mathbb{E}(\lim_{t\to\infty} G_{t\wedge\tau}) = \mathbb{E}(G_\tau)$$

This completes STEP 1.

Step 2: Show $G(\tau_B^0, W_{\tau_B^0}) + H(W_{\tau_B^0}) = F(\tau_B^0)$

Observe that $b(W_{\tau_B^0}) = \tau_B^0$:

$$
\begin{aligned}
G(\tau_B^0, W_{\tau_B^0}) + H(W_{\tau_B^0}) &= \int_0^{\tau_B^0} M(s, W_{\tau_B^0})\ ds + \int_0^{b(W_{\tau_B^0})=\tau_B^0} f(s) - M(s, W_{\tau_B^0})\ ds \\
&= \int_0^{\tau_B^0} f(s)\ ds = F(\tau_B^0)
\end{aligned}
$$

Step 3: Final argumentation

Remember that, as τ_B^0 and τ both solve the SEP, we have that $W_{\tau_B^0} \sim W_\tau$. Further, $G(\tau_B^0, W_{\tau_B^0})$ and $F(\tau_B^0)$ are integrable, therefore

$$H(\tau_B^0) = F(\tau_B^0) - G(\tau_B^0, W_{\tau_B^0})$$

is integrable as well. We conclude

$$\mathbb{E}(H(W_\tau)) = \mathbb{E}(H(W_{\tau_B^0}))$$

On the other hand, we have $\mathbb{E}(G(\tau_B^0, W_{\tau_B^0})) \leq \mathbb{E}(G(\tau, W_\tau))$. To show this, we need to remember that because of our "key lemma" we have that $G(t, W_t)$ is a martingale, and therefore OST brings us $\mathbb{E}(G(\tau_B^0, W_{\tau_B^0})) = \mathbb{E}(G(0, W_0)) = G(0, W_0)$.

But the key lemma (3.13) also makes sure that $G(t \wedge \tau, W_{t\wedge\tau})$ is a submartingale, hence $G(0, W_0) \leq \mathbb{E}(G(t \wedge \tau, W_{t\wedge\tau}))$ for every $t \geq 0$. Therefore:

$$\mathbb{E}(G(\tau_B^0, W_{\tau_B^0})) = G(0, W_0) \leq \lim_{t\to\infty} \mathbb{E}(G(t \wedge \tau, W_{t\wedge\tau})) \overset{\text{STEP 1}}{=} \mathbb{E}(G(\tau, W_\tau))$$

After all, we remember that $G(t, x) + H(x) \leq F(t)$ always holds and finally show:

$$\mathbb{E}(F(\tau_B^0)) \overset{\text{STEP 2}}{=} \mathbb{E}\big(G(\tau_B^0, W_{\tau_B^0}\big) + \mathbb{E}\big(H(W_{\tau_B^0})\big) \leq \mathbb{E}\big(G(\tau, W_\tau\big) + \mathbb{E}\big(H(W_\tau)\big) \leq F(\tau)$$

$\square$

4 Financial Applications of the SEP

In this chapter we will use the theory we established so far to derive some results in the field of Financial Mathematics. Very important for this matter will be Root's solution for the SEP and its optimality property.

Until now, we considered only the classical version of the SEP: Given a Brownian motion (W_t) and a probability measure μ, find a stopping time τ (integrable if possible) such that $W_\tau \sim \mu$. We showed that Root's solution τ_B has the property that for every $t \in \mathbb{R}^+$, for every convex and nondecreasing function F and for every other solution τ we always have

$$F(\tau_B) \leq F(\tau).$$

In this section we need to consider the SEP also for the more general class of processes (X_t) satisfying

$$dX_t = \sigma(X_t)\ dW_t,$$

where $\sigma : \mathbb{R} \to \mathbb{R}$. We assume:

i) $|\sigma(x) - \sigma(y)| \leq K \cdot |x - y|$ for any $x,y \in \mathbb{R}$ (Lipschitz continuity)

ii) $0 < \sigma(x)^2 < K(1 + x^2)$ for every $x \in \mathbb{R}$

iii) σ is smooth

These conditions exclude the function $\sigma(x) = x$ as $\sigma^2(0) = 0$, but it can be shown that all our theory also works for this special case (see [3, Remark 5.5]). Note that Root's approach works for this more general version of the SEP as well (as we did not use any specific property of Brownian motion in the proof). Further, with slight adaptation of the last section, one can show Root's optimality property as well. Interested readers may read Rost [16] for Root's approach in the general context, and Cox & Wang [3] for the proof of optimality.

Remark 4.1.
If we consider the SEP (X, μ), where (X_t) is not a Brownian motion, Wald's Lemma (see Lemma 1.12) does no longer apply. Hence it is possible to get integrable solutions without assuming μ to be centered. We can state a stronger result for the existence of the Root Barrier and the Root solution for a measure μ with support on $(0, \infty)$, which we initially stated only for centered measures (see Theorem 3.11).

One can find this stronger theory in Cox & Wang [3, Section 4]. The reader of this thesis might need to read [3, Section 3] as well to understand why Cox & Wang establish the existence of a solution in this special case. The SEP in Cox & Wang is formulated a little differently, and it is being used that solving the SEP is equivalent to solving the so called *Obstacle problem*.

Studying these more general sections in Cox & Wang is recommended for the reader, but an introduction here would go beyond the scope of this thesis as we would need to define a lot of new terminology. The basic idea and the detailed approach can be seen in section 3.6 for a Brownian motion $X = W$. However, these more general results are essential in the financial applications we are about to show.

4.1 A Lower Bound for Call Prices on Volatility

Let $(P_t)_{t\geq 0}$ be the price process of an asset defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. We make the assumption that (P_t) is continuous. Let $S_t := e^{-rt}P_t$ be the discounted price process, where $r > 0$ is the (constant) interest rate.

Our goal is to derive a model independent bound for the price of the call option

$$([\ln(S)]_T - K)^+$$

with maturity T and strike price K. We will follow HOBSON [7, Section 5.2]. We are going to use the optimality of the Root solution of the SEP (Z, μ), where the process Z is to determine, and μ is the (implied) law of S_T, which we can derive with the formula of Breeden-Litzenberger (see Theorem 2.3).

First of all, note that we have $\ln(S_t) = \ln(P_t) - r \cdot t$, hence $[\ln(S)]_t = [\ln(P)]_t$ since the nondecreasing part $r \cdot t$ does not contribute to quadratic variation. From now on we use the notation $X_t := \ln S_t$.

Note that the price of the underlying call option is $\mathbb{E}_{\mathbb{Q}}\big(([\ln(S)]_T - K)^+\big)$, where $\mathbb{Q}$ is a measure equivalent to $\mathbb{P}$ such that $(\ln(S)_t)$ is a martingale, and that the function $x \mapsto (x-K)^+$ is convex and nondecreasing. This is a very general formula, as we did not use any specific equation for P_t or S_t.

Lemma 4.2.
The following equations are true:

i) $[X]_T = \int_0^T \frac{1}{S_u^2}\, d[S]_u$

ii) $X_T - X_0 = \int_0^T \frac{1}{S_u}\, dS_u - \frac{1}{2}\int_0^T \frac{1}{S_u}\, d[S]_u$

In most literature they write dS_t^2 instead of $d[S]_t$. In this thesis, however, we do not follow this convention because it would be easy to mix up the terms dS_t^2 and $d(S_t^2)$. The equations above can be shown with help of Itô's formula (see Theorem 1.30).

Proof.
The key for the proof of equation i) will be the formula $d[X]_t = d(X_t^2) - 2 \cdot X_t dX_t$, which we already established in 1.35. On the other hand, we know $X_t^2 = \ln(S_t)^2$. Itô's formula for $f(t,x) = \ln(x)^2$ yields

$$d(\ln(S_t)^2) = 2 \cdot \frac{\ln(S_t)}{S_t}\, dS_t + \frac{1}{2}\left(\frac{2}{S_t^2} - \frac{2 \cdot \ln(S_t)}{S_t^2}\right)\, d[S]_t.$$

Further, we need to compute $dX_t = d(\ln(S_t))$:

$$d(\ln(S_t)) = \frac{dS_t}{S_t} - \frac{1}{2 \cdot S_t^2}\, d[S]_t$$

Note that this already shows equation ii). In total we get

$$d[X]_t = 2X_t dX_t \; - \; d(X_t^2)$$
$$= 2X_t\left(\frac{dS_t}{S_t} - \frac{1}{2\cdot S_t^2}\, d[S]_t\right) \; - \; 2\cdot\frac{X_t}{S_t}dS_t + \frac{1}{2}\left(\frac{2}{S_t^2} - \frac{2\cdot X_t}{S_t^2}\right)d[S]_t = \frac{1}{S_t^2}\, d[S]_t,$$

which shows equation i).

□

From now on we assume w.l.o.g $P_0 = S_0 = 1$, which implies $X_0 = 0$. The combination of i) and ii) yields

$$X_t \overset{ii}{=} \underbrace{\int_0^t \frac{1}{S_u} dS_u}_{=:M_t} - \frac{1}{2} \underbrace{\int_0^t \frac{1}{S_u^2} d[S]_u}_{\overset{i}{=}[X]_t}.$$

Now the continuous local martingale (M_t) satisfies $X_t = M_t - \frac{1}{2}[X]_t$, and as the nondecreasing process $[X]_t$ does not contribute to quadratic variation, we even have $[M]_t = [X]_t$. By Dambis-Dubin-Schwarz (see Theorem 3.3) there exists a Brownian motion (W_t) such that we can write

$$M_t = W_{[M]_t} = W_{[X]_t}.$$

Note that we have

$$S_t = e^{X_t} = e^{M_t - \frac{1}{2}[X]_t} = e^{W_{[X]_t} - \frac{1}{2}[X]_t} = Z_{[X]_t}$$

for the process $Z_t = e^{W_t - \frac{1}{2}t}$, which is a local martingale as we already established in example 1.33. Now observe that, as $S_T \sim \mu$, we have that

$$Z_{[\ln(S)]_T} = S_T \sim \mu,$$

hence $[\ln(S)]_T$ solves the SEP for (Z, μ).

Now let us try a different approach: Let τ_B be the Root solution of the SEP for (Z, μ) and let $\tilde{S}_t := Z_{\tau_b \wedge \frac{t}{T-t}}$. This implies $\tilde{S}_T = Z_{\tau_B}$.

By doing the same computations as above, we can show that $[\ln(\tilde{S})]_T$ solves the SEP for (Z, μ). Now τ_B and $[\ln(\tilde{S})]_T$ both are minimal solutions of the SEP (as both are integrable), and we may conclude $\tau_B = [\ln(\tilde{S})]_T$ (see Lemma 3.6) . We call this way of modelling the stock price the **Root Model**.

The price of the underlying call option using the Root Model to describe the behavior of the stock price is $\mathbb{E}_{\tilde{\mathbb{Q}}}\big((\tau_B - K)^+\big)$, where $\tilde{\mathbb{Q}}$ is a measure equivalent to $\mathbb{P}$ such that $(\ln(\tilde{S})_t)$ is a martingale with respect to $\tilde{\mathbb{Q}}$. The optimality property of the Root solution τ_B now yields

$$\mathbb{E}_{\tilde{\mathbb{Q}}}\big((\tau_B - K)^+\big) \leq \mathbb{E}_{\mathbb{Q}}\big(([\ln(S)]_T - K)^+\big)$$

4.2 Finding a Lower Bound by Subhedging the Option

In this section we assume the asset price (P_t) is a stochastic process defined on a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$, and the equation

$$dP_t = P_t \cdot \big(\mu_t \; dt + \sigma_t \; dW_t\big)$$

describes the behavior of P_t. The interest rate process is given by $(r_t)_{t\geq 0}$ and the *discount* is $B_t := e^{\int_0^t r_s \; ds}$. For our theory from the last sections to work in this context, we need r_t and σ_t to be locally bounded and predictable (see Cox & Wang [3, Assumption 6.1]).

By $\mathbb{Q}$ we denote the risk-neutral measure, which is equivalent to $\mathbb{P}$ and defined by Girsanov (see Theorem 1.42) if one uses the *market price of risk process*

$$\theta(t) = \frac{\mu_t - r_t}{\sigma_t}.$$

The Brownian motion yielded by Girsanov (1.42) is (as always in this thesis) denoted by $\tilde{W}_t$, and the corresponding filtration is $\tilde{\mathcal{F}}_t$. Altogether, we have that the discounted price process

$$S_t := \underbrace{e^{-\int_0^t r_s \; ds}}_{=B_t^{-1}} \cdot P_t,$$

satisfies

$$dS_t = S_t \sigma_t \; d\tilde{W}_t,$$

which can be found in Shreve [17, Page 216].

A good intuition for this way of modelling can be achieved if one assumes μ_t and σ_t to be constant, which yields the Black-Scholes model discussed in section 1.6.4. A detailed build up of this model with rigorous explanations for risk-neutral pricing with Girsanov in the case of non constant drift μ_t and volatility σ can be found in [17, Section 5.2.2].

We are going to use the construction from section 3.6 to find a subreplicating portfolio (see Definition 1.39) for an option with payoff

$$F\Big(\int_0^T \sigma_t^2 dt\Big),$$

where F is a convex increasing function with $F(0) = 0$.

To do so, we follow Cox & Wang [3, Chapter 6].

Note that in the current setting we are unable to use theory from section 3.6, which is why we have to perform a time change to get a simpler equation. Rigorous information about this matter can be found in [8, Section 3.4, B]. The basic terminology and the main results are briefly introduced in the appendix (see Section 5.6).

Let us define the time change $C_t := \int_0^t \sigma_s^2 \, ds$, and let

$$A_t := \inf\{s \geq 0 \mid C_s > t\}$$

be the (right-continuous, see Section 5.6) inverse with $C_{A_t} = t$. The time changed Brownian motion is given by

$$\widehat{\tilde{W}}_t = \int_0^{A_t} \sigma_s \, d\tilde{W}_s,$$

the time changed process is $\hat{S}_t = S_{A_t}$ and the time changed filtration is $\hat{\tilde{\mathcal{F}}}_t$ (see [3, Page 886]). Altogether we get

$$\widehat{S}_t = S_{A_t} = \int_0^{A_t} S_u \sigma_u \, d\tilde{W}_u \overset{\text{Le Gall}}{=} \int_0^{A_t} S_u \, d(\underbrace{\sigma \cdot \tilde{W}}_{\int \sigma_u \, d\tilde{W}_u})_t = \big(S \cdot (\sigma\tilde{W})\big)_{A_t} = \big(\widehat{S \cdot (\sigma\tilde{W})}\big)_t \overset{5.18}{=} \big(\widehat{S} \cdot \widehat{\tilde{W}}\big)_t,$$

where we used property (ii) from Le Gall [10, Page 109]. This yields the simplified equation

$$d\widehat{S}_t = \hat{S}_t \, d\widehat{\tilde{W}}_t.$$

Note that we have $\widehat{S}_{C_T} = S_T = B_T^{-1} P_T$, and also $S_{A_t} = \widehat{S}_t$.

Now we are ready to apply results from 3.6 concerning the optimality of solutions of the SEP for $(\widehat{S}, \mathbb{Q}^*)$ (see Cox & Wang [3, Remark 5.5]), where $\mathbb{Q}^*$ is the measure we get from Breeden and Litzenberger (2.3). Let us do a little recap:

Remark 4.3.
In section 3.6 we defined the functions

$$\begin{aligned} M(t,x) &= \mathbb{E}(f(\tau_B^t) \mid \widehat{S}_t = x) \\ Z(x) &= 2 \cdot \int_0^x \int_0^y M(0,z) \, dz \, dy \\ G(t,x) &= \int_0^t M(s,x) \, ds - Z(x) \\ H(x) &= \int_0^{b(x)} \big(f(s) - M(s,x)\big) \, ds + Z(x), \end{aligned}$$

where $f = F'_+$ denoted the right derivative of F and

$$\tau_B^s := \inf\{t \geq s \mid (t, \widehat{S}) \in B\}.$$

Note that we defined them for a Brownian motion W_t instead of considering a more general process like we need here. One can convince themselves that all properties and results still hold by reading [3, Chapter 5]. Remember that F is a convex increasing function and that we need to assume that f is bounded. The inequality we established in *Phase III* (section 3.6) now reads

$$G(t, S_{A_t}) + H(S_{A_t}) = G(t, \widehat{S}_t) + H(\widehat{S}_t) \ \leq F(t) \ = F(C_{A_t}) = F\Big(\int_0^{A_t} \sigma_s^2 \, ds\Big).$$

(see [3, Page 886])

Last but not least, remember that the process $G_t = G(t, \widehat{S}_t)$ is a submartingale, and that $G(t \wedge \tau_B^0, \widehat{S}_{t \wedge \tau_B^0})$ is a martingale. This is similar to the claim from Lemma 3.13, which is only stated for Brownian motion. The more general version of this Lemma we need here is stated and proven in Cox & Wang [3, Lemma 5.2].

The last equality in remark 4.3 tells us that to subreplicate F we can construct a replicating portfolio for $G_t = G(t, \widehat{S}_t)$ and $H(S_T)$. To see this, we write the inequality from the Remark in the following form (using T instead of A_t):

$$F\Big(\int_0^T \sigma_u^2 \, du\Big) = F(C_T) \geq G(C_t, \widehat{S}_{C_T}) + H(\widehat{S}_{C_T}) = G(C_T, S_T) + H(S_T)$$

However, as G_t is a submartingale (as we reminded ourselves in 4.3), subreplication G with a self-financing portfolio is not expected to be possible (see [3, Page 887]), hence we look for a subreplicating portfolio for G.

4.2.1 Subreplication of G

We follow the steps from Cox & Wang [3, Page 887] to find a strategy such that the corresponding portfolio satisfies

$$V_T \leq G(C_T, S_T).$$

By the decomposition theorem of Doob–Meyer (see Theorem 5.5 and read 4.4) we can write the submartingale $G_t = G(t, \widehat{S}_t)$ in the form

$$G_t = M_t + A_t,$$

where (M_t) is a martingale and (A_t) is an increasing process. By the Martingale-Representation Theorem (see Theorem 5.7 and read 4.4) we can write M_t in the form

$$M_t = M_0 + \int_0^t \psi_u \, d\widehat{\tilde{W}}_u.$$

In total we get

$$G_t = \underbrace{G_0}_{A_0+M_0} + \underbrace{A_t - A_0}_{\geq 0} + \underbrace{\int_0^t \psi_u \, d\widehat{\tilde{W}}_u}_{M_t - M_0}.$$

To subreplicate $G_t = G(t, \widehat{S}_t)$, we need to find a process $\widehat{\Phi}$ such that

$$G_t \geq G_0 + \int_0^t \widehat{\Phi}_u \, d\widehat{S}_u.$$

We define

$$\widehat{\Phi}_t = \frac{\Psi_t}{\widehat{S}_t}$$

and compute (after remembering that $d\widehat{S}_t = \widehat{S}_t \, d\widehat{\tilde{W}}_t$):

$$\Psi_u \, d\widehat{\tilde{W}}_u = \Psi_u \, \frac{d\widehat{S}_u}{\widehat{S}_u} = \widehat{\Phi}_u \, d\widehat{S}_u.$$

Altogether, we get

$$G(t, \widehat{S}_t) = G_t \geq G_0 + \int_0^t \psi_u \, d\widehat{\tilde{W}}_u = G_0 + \int_0^t \widehat{\Phi}_u \, d\widehat{S}_u.$$

Remark 4.4.
As we are only interested in times $0 \leq t \leq T$, it suffices that $(G(t \wedge T, \widehat{S}_{t \wedge T}))$ – respectively $(M_{t \wedge T})$ – satisfy the conditions of Doob–Meyer – respectively Martingale Representation Theorem. We can use Doob-Meyer's decomposition theorem, because the collection $(G(t, \widehat{S}_t))_{0 \leq t \leq T}$ is uniformly integrable, which follows from its L^1 convergence and from 5.11. One might note that we don't know whether or not (M_t) is square integrable. Hence, we can not apply the version of the Martingale Representation Theorem stated in the appendix. However, we may apply a stronger version from Karatzas & Shreve [8, Page 170].

The only problem is that the just derived subhedging strategy only works for the time changed discounted stock price $\widehat{S}_t$. In our original model we need to use the strategy

$$\Phi_t := \widehat{\Phi}_{C_t}.$$

We now have

$$G(C_t, S_t) = G(C_t, \widehat{S}_{C_t}) \geq G(0, \widehat{S}_0) + \int_0^{C_t} \widehat{\Phi}_u \; d\widehat{S}_u \overset{5.18}{=} G(0, \underbrace{S_0}_{=\widehat{S}_0}) + \int_0^t \Phi_u \; dS_u$$

So let us consider the portfolio where we hold $\Phi_s \cdot B_T^{-1}$ units of the risky asset and make and investment of $G(0, S_0) \cdot B_T^{-1} - \Phi_0 \cdot P_0 \cdot B_T^{-1}$ in the risk free asset (at time 0). This makes a total of $G(0, S_0) \cdot B_T^{-1}$ of initial investment in the portfolio consisting of the risky and the riskless asset.

Note that we can choose our strategy H^0 for the riskless asset such that the strategy $(H^0, \Phi \cdot B_T^{-1})$ is self-financing (see Remark 1.52). The same computation as in the proof of Lemma 1.44 yields:

$$d(\underbrace{B_t^{-1} \cdot V_t}_{\tilde{V}_t}) = B_T^{-1} \cdot \Phi_t \; d(\underbrace{B_t^{-1} P_t}_{S_t}).$$

It follows that

$$\begin{aligned} V_T &= B_T \cdot \Bigg(\underbrace{V_0 \cdot B_0^{-1}}_{\text{total initial investment}} + \int_0^T B_T^{-1} \Phi_u \; d(\underbrace{B_u^{-1} P_u}_{S_u}) \Bigg) = G(0, S_0) + \int_0^T \Phi_u \; dS_u \\ &\leq G(C_T, S_T), \end{aligned}$$

which shows that the self-financing strategy of holding $(\Phi_t \cdot B_T^{-1})$ units of the risky asset and initially investing $G(0, S_0) \cdot B_T^{-1} - \Phi_0 \cdot P_0 \cdot B_T^{-1}$ in the riskless asset is indeed the strategy we were looking for.

4.2.2 Replication of H

To replicate H with the method from COX & WANG [3] we need to make the assumption that the measure

$$dH'\big((x, y)\big) = H'(y) - H'(x)$$

exists. This can be achieved for instance by assuming that H is a convex function. From now on, we assume that H is a piecewise linear function. This can be justified by arguing that H can always be approximated by such a function in practice (see COX & WANG [3, Page 888]). Further, it is reasonable to assume that H has only finitely many *kinks* K, i.e points where left and right derivative of H do not coincide (see figure 8).

We can write $H(x)$ as follows:

$$H(x) = H(P_0) + H'(P_0) \cdot (x - P_0) + \sum_{K > P_0} (x - K)^+ \cdot H'(dK) + \sum_{0 < K < P_0} (K - x)^+ \cdot H'(dK)$$

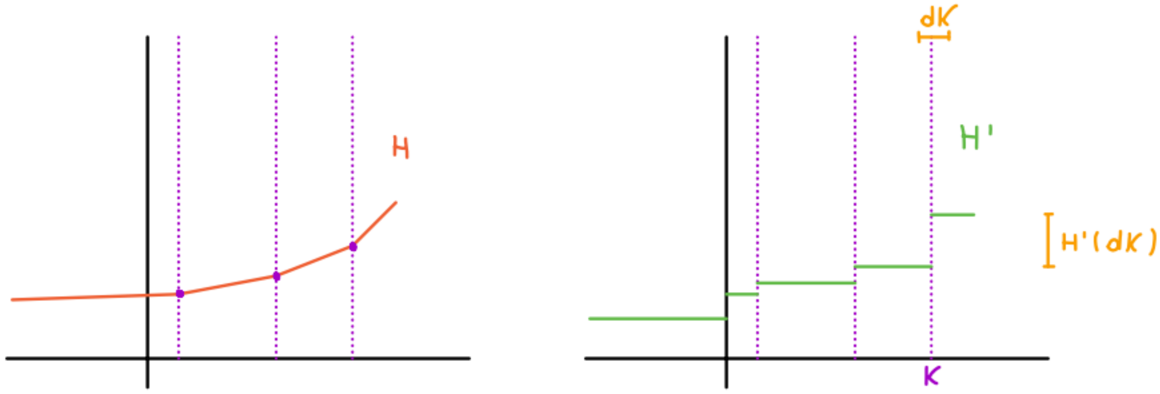


Figure 8: A possible graph of the convex and piecewise linear function H and its derivative H', which does not exist at the highlighted points.

One can get intuition for this formula in figure 9. Note that this only holds in the case where P_0 is not a *kink*, because otherwise the term $H'(P_0)$ would not exist. However, if P_0 is a *kink* we can use $H'(P_0 - \varepsilon)$ (respectively $H'(P_0 + \varepsilon)$ if $x < P_0$) instead. This fact becomes also clear by looking at figure 9. For simplicity let us assume that $H'(P_0)$ exists.

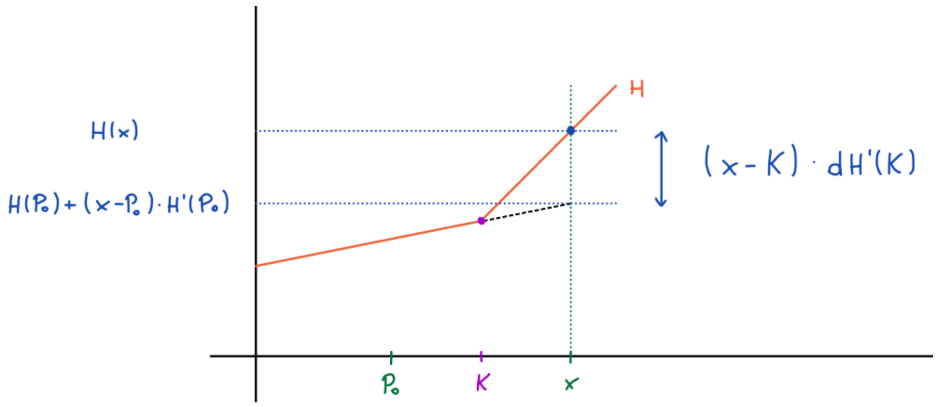


Figure 9: A graphical way of explaining the formula for $H(x)$.

Now let us take $x = S_T = B_T^{-1} \cdot P_T$:

$$H(S_T) = \underbrace{H(P_0) + H'_+(P_0) \cdot (B_T^{-1} \cdot P_T - P_0)}_{H(P_0) - H'_+(P_0) \cdot P_0 + H'_+(P_0) \cdot B_T^{-1} P_T} + \sum_{K > P_0} \underbrace{(B_T^{-1} \cdot P_T - K)^+}_{B_T^{-1} \cdot (P_T - B_T K)^+} \cdot H'(dK) + \sum_{0 < K < P_0} \underbrace{(K - B_T^{-1} \cdot P_T)^+}_{B_T^{-1} \cdot (B_T K - P_T)^+} \cdot H'(dK)$$

Altogether, we can replicate $H(S_T)$ by a portfolio which consists of ..

- ..an amount of $B_T^{-1} \cdot \big(H(P_0) - H'_+(P_0) \cdot P_0\big)$ of an riskless asset growing with interest rate r_t, for instance money on a bank account. At time T this amount will be $H(P_0) - H'_+(P_0) \cdot P_0$.
- ..$B_T^{-1} \cdot H'_+(P_0)$ units of the risky asset (whose price is being modelled with P)
- ..$B_T^{-1} \cdot H'(dK)$ units of any call option with a strike price $B_T \cdot K$, where $K > P_0$ is a *kink*.
- ..$B_T^{-1} \cdot H'(dK)$ units of any put option with strike price $B_T \cdot K$, where $0 < K < P_0$.

(So you need to buy as many options as H has kinks.)

After these preparations we are ready for the *final theorem* of this thesis. The basic idea now is that we constructed a subreplicating portfolio for our option F, and the price of the option can not be less than the price of the subreplicating portfolio without causing arbitrage.

Remark 4.5.
To acquire the subreplicating portfolio of the option with payoff $F\big(\int_0^T \sigma_u^2 \, du\big)$ we just constructed, we..

i) ..buy a total of $B_T^{-1} \cdot \big(\Phi_0 + H'(P_0)\big)$ of the risky asset at time 0, which costs us

$$\underbrace{B_T^{-1} \cdot \Phi_0 \cdot P_0}_{\text{for G}} \; + \; \underbrace{B_T^{-1} \cdot H'(P_0) \cdot P_0}_{\text{for H}}.$$

ii) ..invest an amount of

$$\underbrace{B_T^{-1} \cdot G(0, P_0)}_{\text{for G}} - \underbrace{B_T^{-1} \cdot \Phi_0 \cdot P_0}_{\text{for G}} + \underbrace{B_T^{-1} \cdot H(P_0)}_{\text{for H}} - \underbrace{B_T^{-1} \cdot H'(P_0) \cdot P_0}_{\text{for H}}$$

into the riskless asset.

iii) ..buy $B_T^{-1} \cdot H'(dK)$ units of the call option of strike price $B_T \cdot K$ for any *kink* $K > P_0$. This costs

$$B_T^{-1} \cdot \sum_{K > P_0, \text{ K kink}} \mathcal{C}(B_T K) \cdot H'(dK),$$

where $\mathcal{C}$ denotes the price of the call option.

iv) ..buy $B_T^{-1} \cdot H'(dK)$ units of the put option of strike price $B_T \cdot K$ for any *kink* $0 < K < P_0$. This costs

$$B_T^{-1} \cdot \sum_{0 < K < P_0, \text{ K kink}} \mathcal{P}(B_T K) \cdot H'(dK),$$

where $\mathcal{P}$ denotes the price of the put option.

The total cost of this portfolio is the sum of all these terms, i.e

$$\begin{aligned}\text{total cost} &= B_T^{-1} \cdot G(0, P_0) + B_T^{-1} \cdot H(P_0) \\ &\quad + B_T^{-1} \cdot \sum_{K>P_0,\ \text{K kink}} \mathcal{C}(B_T K) \cdot H'(dK) + B_T^{-1} \cdot \sum_{0<K<P_0,\ \text{K kink}} \mathcal{P}(B_T K) \cdot H'(dK).\end{aligned}$$

Assume price of the underlying option with payoff $F\big(\int_0^T \sigma_u^2\ du\big)$ is smaller than this bound. Then we can find arbitrage by proceeding as in section 1.6.2:
Go short on the subreplicating portfolio, i.e borrow it at time 0 and sell it immediately for the price we just calculated. By our assumption, the benefit of this sell is big enough to buy the option with payoff $F\big(\int_0^T \sigma_u^2\ du\big)$. At time T, this option yields more than the subreplicating portfolio. We use the money from this yield at time T to buy the subreplicating portfolio, and give it back.

4.2.3 The Final Theorem

The following theorem can be found in COX & WANG [3, Theorem 6.4]:

Theorem 4.6.
Let F be a convex, increasing function with $F(0) = 0$ and bounded right derivative $f(t) = F'_+(t)$. If the price of the option with payoff

$$F\Big(\int_0^T \sigma_u^2\ du\Big)$$

is less than

$$\begin{aligned}&B_T^{-1} \cdot G(0, P_0) + B_T^{-1} \cdot H(P_0) \\ &+ B_T^{-1} \cdot \sum_{K>P_0,\ \text{K kink}} \mathcal{C}(B_T K) \cdot H'(dK) + B_T^{-1} \cdot \sum_{0<K<P_0,\ \text{K kink}} \mathcal{P}(B_T K) \cdot H'(dK)\end{aligned}$$

there exists arbitrage. The functions G and H are defined as in Remark 4.3, $\mathcal{C}(.)$ is the price of a call option and $\mathcal{P}(.)$ is the price of a put option.

Proof.
Let us consider the SEP for $(\widehat{S}_t, \mathbb{Q}^*)$, where $(\widehat{S}_t)$ is the time-changed discounted asset price, and $\mathbb{Q}^*$ is the measure obtained from Breeden & Litzenberger (see Theorem 2.3) by using Call Prices relying on the original asset price process (P_t).

The existence of a Root solution is explained in Remark 4.1, hence the constructions from the sections 4.2.1 and 4.2.2 are well defined and yield a subreplicating portfolio.

If the price of the option is smaller than the given bound, we proceed as in remark 4.5 to generate arbitrage.

Remark 4.7.
The bound from theorem 4.6 is optimal in the following sense:
There exists an arbitrage-free model under which the price of the option is this bound.
One can find a proof for this claim in Cox & Wang [3, Page 889].

5 Appendix

5.1 Convergence Theorems from Measure Theory

In this section we use [1][Bogachev]. Note that we formulate the following theorems (unlike [1]) in the context of probability theory: Instead of using the notation $\int_\Omega X \, d\mathbb{P}$ we already write $\mathbb{E}(X)$. Further, we assume the functions X_n and X are random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ instead of stating the theorems for general measurable functions.

Theorem 5.1 (Dominated Convergence Theorem, DCT, [1], p.130).
Let (X_n) be a sequence of integrable random variables and let $\lim_{n\to\infty} X_n = X$ hold almost everywhere. Suppose there exists an integrable random variable Y such that for every n we have

$$|X_n| \leq Y \quad a.e.$$

Then X is integrable and

$$\lim_{n\to\infty} \mathbb{E}(X_n) = \mathbb{E}(X).$$

Theorem 5.2 (Monotone Convergence Theorem, MCT, [1] p.130).
Let (X_n) be a sequence of integrable random variables satisfying $X_n \leq X_{n+1}$ a.s for all $n \in \mathbb{N}$. Suppose that

$$\sup_{n\in\mathbb{N}} \mathbb{E}(X_n) \ < \infty.$$

Then, the function $X := \lim_{n\to\infty} X_n$ is integrable and finite almost everywhere, and we have

$$\lim_{n\to\infty} \mathbb{E}(X_n) = \mathbb{E}(X).$$

Lemma 5.3 (Fatou's Lemma 1, [1], p.131).
Let (X_n) be a sequence of nonnegative integrable random variables, and suppose $\lim_{n\to\infty} X_n = X$ holds almost everywhere. Further, suppose that

$$\sup_{n\in\mathbb{N}} \mathbb{E}(X_n) \leq K < \infty.$$

Then the random variable X is integrable and

$$\mathbb{E}(X) \leq K.$$

Further we have

$$\mathbb{E}(X) \leq \liminf_{n\to\infty} \mathbb{E}(X_n).$$

Lemma 5.4 (Fatou's Lemma 2, [1], p.132).
Let (X_n) be a sequence of nonnegative integrable random variables, and suppose

$$\sup_{n\in\mathbb{N}} \mathbb{E}(X_n) \leq K < \infty.$$

Then, the function $\liminf_{n\to\infty} X_n$ is integrable, and one has

$$\mathbb{E}(\liminf_{n\to\infty} X_n) \leq \liminf_{n\to\infty} \mathbb{E}(X_n) \leq K$$

5.2 Doob-Meyer Decomposition

In this section we shortly present the Doob-Meyer Decomposition Theorem as in Karatzas & Shreve [8, Page 24, Theorem 4.10], but in a little less general form. We are going to avoid the more general setting of the classes D and DL.

Theorem 5.5.
Let (X_t) be a right-continuous submartingale with respect to a filtration $(\mathcal{F}_t)$, which is right-continuous (see Definition 5.15) and $\mathcal{F}_0$ contains all events with probability 0. Assume (X_t) is uniformly integrable. Then we can write X_t in the form

$$X_t = M_t + A_t,$$

where (M_t) is a right-continuous martingale with respect to $(\mathcal{F}_t)$ and (A_t) is an increasing process. This decomposition is unique.

5.3 Martingale Representation Theorem

Definition 5.6 ([10], page 42).
Let $(X_t)_{t\geq 0}$ be a stochastic process. The **natural** (or sometimes canonical) filtration of X is defined by $\mathcal{F}_t := \sigma(X_s \mid 0 \leq s \leq t)$ and $\mathcal{F}_\infty := \sigma(X_s \mid s \geq 0)$.

In the following theorem let W_t be a Brownian motion with natural filtration $\mathcal{F}_t$:

Theorem 5.7 (Martingale Representation Theorem, [9], 4.2.4).
Let (M_t) be a martingale w.r.t the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$, and let M be square integrable, i.e for every t we have $M_t \in L^2(\Omega, \mathcal{F}, \mathbb{P})$. Then there exists a previsible process $(H_t)_{0\leq t\leq T}$ such that

$$\mathbb{E}\Big(\int_0^T H_s^2\, ds\Big) < \infty$$

and for all $t \in [0, T]$ we have:

$$M_t = M_0 + \int_0^T H_s \, dW_s$$

5.4 Leibniz' Rule and Differentiation of Limits

In this section we state the two theorems we need in the proof of Breeden-Litzenberger. Leibniz' Rule can be found in Heuser [6, page 101].

Lemma 5.8 (Leibniz' Rule).
Let $f : (\alpha, \beta) \times (c, d) \to \mathbb{R}$, $(t, x) \mapsto f(t, x)$ be a continuous function with continuous partial derivative $\frac{\partial}{\partial t} f(t, x)$. Let a and b be functions $(\alpha, \beta) \to (c, d)$ be differentiable with continuous derivative. Then, the function

$$F(t) = \int_{a(t)}^{b(t)} f(t, x) \, dx$$

is differentiable in (α, β), and we have

$$F'(t) = f\big(t, b(t)\big) \cdot b'(t) \; - \; f\big(t, a(t)\big) \cdot a'(t) \; + \; \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(t, x) \, dx$$

The following theorem is from Forster [5, page 270]:

Theorem 5.9.
Let $f_n : [a, b] \to \mathbb{R}$ be continuously differentiable for all n, and let f_n converge pointwise against f. Suppose f_n' converges uniformly. Then f is differentiable and

$$f'(x) = \lim_{n \to \infty} f_n'(x).$$

5.5 Martingale Convergence, Jensen and Doob's Inequality

Theorem 5.10 ([10], Proposition 3.15, Page 53).
Let (X_t) be a martingale with right-continuous sample paths. Then, for every $t > 0$ and every $p > 1$ we have

$$\mathbb{E}\big(\sup_{0 \le s \le t} |X_s|^p\big) \le \big(\frac{p}{p-1}\big)^p \cdot \mathbb{E}(|X_t|)^p.$$

Theorem 5.11 ([10], Theorem 3.19, Page 58).
Let (X_t) be a supermartingale with right-continuous sample paths. Assume that the collection $(X_t)_{t\geq 0}$ is bounded in L^1. Then there exists a random variable $X_\infty \in L^1$ such that

$$\lim_{t\to\infty} X_t = X_\infty \quad \text{a.s.}$$

Definition 5.12 ([10], Definition 3.20, Page 59).
A martingale (X_t) is called **closed**, if there exists a random variable $Z \in L^1$ such that for every $t \geq 0$ we have

$$X_t = \mathbb{E}(Z \mid \mathcal{F}_t).$$

Theorem 5.13 ([10], Theorem 3.21, Page 59).
Let (X_t) be a martingale with right-continuous sample paths. Then the following properties are equivalent:

i) (X_t) is a closed martingale.

ii) The random variables $(X_t)_{t\geq 0}$ are uniformly integrable.

iii) The martingale (X_t) converges a.s and in L^1 as $t \to \infty$.

If one of these properties holds (i.e all of them hold), then we also have $X_t = \mathbb{E}(X_\infty \mid \mathcal{F}_t)$ for every $t \geq 0$, where $X_\infty(\omega) := \lim_{t\to\infty} X_t(\omega)$.

Theorem 5.14 (Jensen, [17], Page 70).
Let the random variable X be defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G}$ be a sub sigma algebra of $\mathcal{F}$. If $\varphi(x)$ is a convex function and X is integrable, then

$$\mathbb{E}\big(\varphi(X) \mid \mathcal{G}\big) \geq \varphi\big(\mathbb{E}(X \mid \mathcal{G})\big)$$

5.6 Time-Changed Processes

In this section we briefly introduce the theory of time-changes. As reference we use Revuz & Yor [13, chapter V, §1].

Definition 5.15 ([10], page 42).
We call a filtration $(\mathcal{F}_t)_{t\geq 0}$ **right continuous**, if for all $t \geq 0$ we have

$$\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$$

Definition 5.16 ([13], (1.2)).
A **time change** C is a family $(C_s)_{s\geq 0}$ of stopping times such that $s \mapsto C_s$ is a.s increasing and right continuous.

In this section, let us always assume the filtration $\mathcal{F}_t$ is right-continuous, and that $(A_t)_{t\geq 0}$ is an increasing, right-continuous (i.e sample paths are right-continuous), to $(\mathcal{F}_t)$ adapted process. We define

$$C_s := \inf\{t \geq 0 \mid A_t > s\}.$$

One can show that C_s defines a time change (see [13, (1.1)]). Further, given a time change C we get an increasing right-continuous process by defining

$$A_t := \inf\{s \geq 0 \mid C_s > t\}$$

If X_t is a previsible process w.r.t $\mathcal{F}_t$, then the process $\hat{X}_t := X_{C_t}$ is an $\hat{\mathcal{F}}_t$-adapted process. We call $\hat{X}$ the **time-changed process**.

Note that, as C_s is an increasing process, the limit $C_{s^+} := \lim_{u\nearrow s} C_u$ exists, and

$$C_{s^-} : \inf\{t \geq 0 \mid A_t \geq s\}$$

Definition 5.17 ([13], (1.3)).
We call a process X **continuous with respect to a time change** C, if X is constant on each interval $[C_{t^-}, C]$.

In this thesis, the following proposition will be important. It tells us that if we have processes X and H it does not make a difference whether we apply time changes and take the stochastic integral $(\hat{X} \cdot \hat{H})$, or first take the stochastic integral $(H \cdot X)$ and then apply the time change $(\widehat{X \cdot H})$.

Theorem 5.18 ([13], (1.4)).
Let H_t be previsible with respect to $\mathcal{F}_t$, then $\hat{H}_t$ is previsible with respect to $\hat{\mathcal{F}}_t$. Further, suppose X has finite variation and is continuous with respect to C. Then

$$(\hat{X} \cdot \hat{H}) = (\widehat{X \cdot H}),$$

or more precisely

$$\int_{C_0}^{C_t} H_s \, dX_s = \int_0^t 1_{C_u < \infty} H_{C_u} \, dX_{C_u}.$$

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